CHAPTER 10

Cohomology

1. Cohomology

As you learned in linear algebra, it is often useful to consider the *dual* objects to objects under consideration. This principle applies much more generally. For example, in order to understand a differentiable manifold or algebraic variety, it is useful to study the appropriate functions on it. In algebraic topology, a similar idea is frutiful. Instead of considering chains which are linear combinations of simplices or cells, we consider functions on simplices or cells. The resulting theory is called *co*homology, and it is dual to homology. For us the most important aspect of cohomology theory is that under appropriate circumstances it gives us a *ring* structure. This allows for more use of algebraic techniques in solving geometric problems.

Let X be a space, and let M be an abelian group. Denote by $S^n(X; M)$ the set of all functions defined on the set of singular n-simplices of X with values in M. Any such function defines a unique homomorphism $f : S_n(X) \to M$ and conversely any such homomorphism defines such a function. Hence, we may also write

$$S^n(X; M) = \mathrm{Hom}(S_n(X), M).$$

We shall show how to make S^n into the analogue of a chain complex, but it is worthwhile doing that in a somewhat more general context.

1.1. Some Homological Algebra. A *cochain complex* C^* is a collection of abelian groups $C^n, n \in \mathbf{Z}$ and homomorphisms $\delta^n : C^n \to C^{n+1}$ such that $\delta^{n+1} \circ \delta^n = 0$ for each n. A cochain complex is called non-negative if $C^n = \{0\}$ for $n < 0$. We shall consider only non-negative cochain complexes unless otherwise states.

The easiest way to construct a cochain complex is from a chain complex. Namely, let S_* denote a chain complex, and let M be an abelian group. Let

$$C^n = \mathrm{Hom}(S_n, M),$$

and let $\delta^n = \mathrm{Hom}(\partial_{n+1}, \mathrm{Id})$, i.e.,

$$\delta^n(f) = f \circ \partial_{n+1} \qquad f \in \mathrm{Hom}(S_n, M).$$

Here $\text{Hom}(M, N)$ denotes the set of homomorphisms from one abelian group into another. This is the object part of a functor into the category of abelian groups. Given, $i : M \to M', j : N \to N'$, we also have the homomophism $\text{Hom}(i, j) : \text{Hom}(M', N) \to \text{Hom}(M, N')$ defined by

$$\text{Hom}(i, j)(f) = j \circ f \circ i \qquad f \in \text{Hom}(M', N).$$

Note that his functor is *contravariant* in the first variable, and covariant in the second, i.e.,

$$\text{Hom}(i_1 \circ i_2, j_1 \circ j_2) = \text{Hom}(i_2, j_1) \circ \text{Hom}(i_1, j_2).$$

(You should draw some diagrams and check for yourself that everything makes sense and that the rule is correct.)

The functor Hom preserves finite direct sums, i.e.,

$$\text{Hom}(\bigoplus_i M_i, N) \cong \bigoplus_i \text{Hom}(M_i, N)$$

$$\text{Hom}(M_i, N) \cong \bigoplus_i \text{Hom}(M, N_i).$$

The second statement in fact holds for any direct sums, but the first statement only holds for finite direct sums. The first statement follows from the fact that any homomorphism $f : \bigoplus_i M_i \to N$ is completely determined by its restrictions f_i to the summands M_i. Moreover, the correspondence

$$f \leftrightarrow (f_i)_i$$

actually provides an isomorphism of abelian groups. Similar remarks apply to the second isomorphism.

Another important fact about Hom is the relation

$$\text{Hom}(\mathbf{Z}, N) \cong N$$

the isomorphism being provided by $f \in \text{Hom}(Z, N) \mapsto f(1)$ and $a \in N \mapsto f_a$ where $f_a(n) = na$. (This is analogous to the isomorphism $M \otimes \mathbf{Z} \cong M$.)

The functor Hom is *left exact*. That means the following

PROPOSITION 10.1. *Let $0 \to M' \to M \to M'' \to 0$ be a ses of abelians groups and let N be an abelian group. Then*

$$0 \to \text{Hom}(M'', N) \to \text{Hom}(M, N) \to \text{Hom}(M', N)$$

is exact. Similarly, if $0 \to N' \to N \to N'' \to 0$ is exact then

$$0 \to \text{Hom}(M, N') \to \text{Hom}(M.N) \to \text{Hom}(M, N'')$$

is exact.

In fact, in these assertions, we may drop the assumption that the 'M' sequence is exact on the left or that the 'N' sequence is exact on the right.

Note the reversal of arrows in the first statement. In effect this says that Hom preserves kernels for the second variable and sends cohernels to kernels for the first variable.

PROOF. You just have to check what each of the assertions means. We leave it as an exercise for the student. □

It is easy to see that Hom is not generally exact. For example, $0 \to Z \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ and $N = \mathbf{Z}$ yields

$$0 \to \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}) = 0 \to \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z} \xrightarrow{2} \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}$$

and the last homomophism is not onto. We shall return to the question of when Hom is exact later.

Returning to our study of cochain complex, note that $C^* = \mathrm{Hom}(S_*, M)$ is a cochain complex because

$$\delta^{n+1} \circ \delta^n = \mathrm{Hom}(\partial_{n+2}, \mathrm{Id}) \circ \mathrm{Hom}(\partial_{n+1}, \mathrm{Id}) = \mathrm{Hom}(\partial_{n+1} \circ \partial_{n+2}, \mathrm{Id}) = 0.$$

We may in fact sumbsume the theory of cochain complexes in that of chain complexes as follows. Given a cochain complex C^*, define $C_n = C^{-n}$ and let $\partial_n : C_n \to C_{n-1}$ be $\delta^{-n} : C^{-n} \to C^{-n+1} = C^{-(n-1)}$. Note that non-negative cochain complexes correspond exactly to non-positive chain complexes. Hence, the notions of cycles, boundaries, homology, chain homotopy, etc. all make sense for cochain complexes. However, we shall prefix everything by 'co', and use the appropriate superscript notation when discussing cochain complexes. (Note however, that all the arrows go in the 'opposite' direction when using superscript notation.) In particular, if C^* is a cochain complex, then

$$H^n(C^*) = Z^n(C^*)/B^n(C^*) = \mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1})$$

is the nth *cohomology* group of the complex.

As above, let X be a topological space, M an abelian group, and define

$$H^*(X; M) = H^*(\mathrm{Hom}(S_*(X), M)).$$

Because of the above remarks, it is easy to see that this is a functor both on topological spaces X and also on abelian groups M. However, it is contravariant on topological spaces. In more detail, if $f : X \to Y, g : Y \to Z$ are maps, then we have

$$S_*(g \circ f) = S_*(g) \circ S_*(f) : S_*(X) \to S_*(Z)$$

and

$$S^*(g\circ f,\mathrm{Id})=\mathrm{Hom}(g\circ f,\mathrm{Id})=\mathrm{Hom}(f,\mathrm{Id})\circ\mathrm{Hom}(g,\mathrm{Id}):$$
$$\mathrm{Hom}(S_*(Z),M)\to\mathrm{Hom}(S_*(Y),M)\to\mathrm{Hom}(S_*(X),M).$$

Hence,

$$H*(g\circ f;\mathrm{Id})=H^*(f;\mathrm{Id})\circ H^*(g\circ;\mathrm{Id}):$$
$$H^*(Z;M)\to H^*(Y;M)\to H^*(X;M).$$

PROPOSITION 10.2. *Let X be a topological space with finitely many components, M an abelian group. Then*

$$H^0(X;M)=\bigoplus_{comps\ of\ X}M$$

PROOF. We have by definition, an exact sequence

$$S_1(X)\xrightarrow{\partial_1}S_0(X)\to H_0(X)\to 0$$

Hence,

$$0\to\mathrm{Hom}(H_0(X),M)\to\mathrm{Hom}(S_0(X),M)\xrightarrow{\delta^0}\mathrm{Hom}(S_1(X),M)$$

is exact. The result now follows from the fact that $H_0(X)$ is free on the components of X. □

What do you think the result should be if X has infinitely many components?

We may repeat everything we did for singular homology except that the arrows all get turned around. Thus, we define

$$\tilde{H}_0(X;M)=\mathrm{Coker}(H^0(\{P\};M)=M\to H^0(X;M)).$$

This may also be dscribed by taking the homology of the cochain complex

$$\mathrm{Hom}(\tilde{S}_n(X),M)$$

Suppose now that A is a subspace of X. Consider the short exact sequence of of chain complexes

$$0\to S_*(A)\to S_*(X)\to S_*(X,A)\to 0.$$

Since $S_n(X,A)$ is free, it follows that the above sequence yields a *split* short exact sequence of abelian groups for each n. (However, the splitting morphisms $S_n(X,A)\to S_n(X)$ won't generaly constitute a chain map!) Since Hom preservers direct sums (and also splittings), it follows that

$$0\to\mathrm{Hom}(S_*(X,A),M)\to\mathrm{Hom}(S_*(X),M)\to\mathrm{Hom}(S_*(A),M)\to 0$$

is an exact seqeunce of chain complexes. (It also splits for each n, but it doesn't necessarily split as a sequence of cochain complexes.) Define

$$H^*(X, A; M) = H^*(\mathrm{Hom}(S_*(X, A), M).$$

Then we get connecting homomorphisms

$$\delta^n : H^n(A; M) \to H^{n+1}(X, A; M)$$

such that

$$\to H^n(X, A; M) \to H^n(X; M) \to H^n(A; M) \to H^{n+1}(X, A; M) \to \dots$$

is exact. As in the case of homology, there is a similar sequence for reduced cohomology.

The homotopy axiom holds because the chain homotopies induce 'cochain homotopies' of the releveant cohaing complexes.

Excision holds, i.e., if $\overline{U}$ is contained in the interior of A, then $H^*(X, A; M) \to H^*(X - U, A - U; M)$ is an isomorphism. (Of course, the direction of the homomorphism is rversed.

Finally, there is a Mayer–Vietoris sequence in cohomology which looks exactly like the one in homology except that all the arrows are reversed.

Note that cohomology always requires a coefficient group. Hence, it is analagous to singular cohomology with coefficients. As in that theory, we also get a long exact sequence from coefficient sequences.

Proposition 10.3. *Let $0 \to N' \to N \to N'' \to 0$ be a s.e.s. of abelian groups. There is a natural connecting homomoprhism $H^n(X; N'') \to H^{n+1}(X; N')$ such that the long sequence*

$$\to H^n(X; N') \to H^n(X; N'') \to H^n(X; N'') \to H^{n+1}(X; N') \to \dots$$

is exact. (There is also a similar sequence for reduced cohmology.)

Proof. Apply the following lemma to obtain the s.e.s

$$0 \to \mathrm{Hom}(S_*(X), N') \to \mathrm{Hom}(S_*(X), N) \to H_*(S_*(X), N'') \to 0.$$

Lemma 10.4. *Let M be free. Then for each ses $0 \to N' \to N \to N'' \to 0$, the sequence*

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0$$

is exact.

Proof. We need only show that $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'')$ is an epimorphism. But this amounts to showing that every homomorphism $f'' : M \to N''$ may be lifted to a homomorphism $f : M \to N$.

However, this follows because $N \to N''$ is an epimorphism and M is free. □

□

2. The Universal Coefficient Theorem

One might be tempted to think that cohomology is simply dual to homology, but that is not generally the case. There is a universal coefficient theorem analagous to that for homology with coefficients.

Let C_* be a chain complex. Define a homomophism

$$\alpha : H^n(\mathrm{Hom}(C_*, N) \to \mathrm{Hom}(H_n(C_*), M)$$

by

$$\alpha(\overline{f})(\overline{z}) = f(z)$$

for f an n-cocycle in $\mathrm{Hom}(C_n, M)$ and z a cycle in C_n. We leave it to the student to check that this is well defined, i.e., it depends only on the cohomology class of f and the homology class of z.

We shall show that if C_* is free, then α is always onto, and in good cases it is an isomorphism. To this end, consider as in the case of homology the short exact sequence of chain complexes

$$0 \to Z_*(C_*) \to C_* \to B_{+,*} \to 0.$$

These split in each degree, so

$$0 \to \mathrm{Hom}(B_{+,*}, N) \to \mathrm{Hom}(C_*, N) \to \mathrm{Hom}(Z_*, N) \to 0$$

is an exact sequence of cochain complexes. (The coboundaries on either end are trivial.) Hence, this induces a long exact sequence in cohomology

$$\to \mathrm{Hom}(B_{n-1}, N) \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Hom}(Z_n, N) \xrightarrow{\Delta^n} \mathrm{Hom}(B_n, N) \to$$

Note the shift in degree relating $B_{+,n}$ to B_{n-1}. This yields short exact seqeunces

$$0 \to \mathrm{Coker}\,\Delta^{n-1} \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Ker}\,\Delta^n \to 0.$$

It is not hard to check that $\Delta^n : \mathrm{Hom}(Z_n, N) \to \mathrm{Hom}(B_n, N)$ is just dual to the inclusion $i_n : B_n \to Z_n$. From the short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0$$

and the left exactness of Hom, it follows that

$$0 \to \mathrm{Hom}(H_n, N) \to \mathrm{Hom}(Z_n, N) \to \mathrm{Hom}(B_n, N) \tag{44}$$

is exact, so we may identify $\operatorname{Ker}\Delta^n$ with $\operatorname{Hom}(H_n, N)$. Hence, we get an epimorphism

$$H^n(\operatorname{Hom}(C_n, N)) \to \operatorname{Hom}(H_n(C_*), N)$$

which it is easy to check is just the homomorphism α defined above.

It remains to determine $\operatorname{Coker}\Delta^{n-1}$. Shifting degree up by one, we see we need to find the cokernel of the homomophism on the right of 44. Of course, the best case will be when that cokernel is trivial, i.e., when the functor Hom is exact on the right as well as on the left for that particular sequence.

Consider now the general problem the above discussion suggests. Let M be an abelian group and let $0 \to P_1 \to P_0 \to M \to 0$ provide a free presentation as in our discussion of Tor. Consider the cochain complex $\operatorname{Hom}(P, N)$ and define a bifunctor $\operatorname{Ext}(M, N)$ such that

$$\operatorname{Ext}(M, N) \cong H^1(\operatorname{Hom}(P, N)) = \operatorname{Coker}\{\operatorname{Hom}(P_0, N) \to \operatorname{Hom}(P_1, N)\}.$$

The theory at this point proceeds in a completely analagous fashion to the theory of Tor with some important exceptions. The functor $\operatorname{Ext}(M, N)$ is well defined. It is contravariant in the first argument (reverses arrows) and covariant in the second argument. It preserves *finite* direct sums. It is not generally commutative, i.e., there is no particular relation between $\operatorname{Ext}(M, N)$ and $\operatorname{Ext}(N, M)$ since there is no such relation for Hom. Because of the lack of commutativity, some of the proofs which relied on commutativity in the case of Tor have to be recast.

One very important property of Ext is the following

PROPOSITION 10.5. *Let $0 \to M' \to M \to M'' \to 0$ be exact. Then there is a natural connecting homomorphism such that the sequence*

$$\begin{aligned} 0 \to \operatorname{Hom}(M'', N) \to \operatorname{Hom}(M.N) \to \operatorname{Hom}(M', N) \to \\ \operatorname{Ext}(M'', N) \to \operatorname{Ext}(M, N) \to \operatorname{Ext}(M', N) \to 0 \end{aligned}$$

is exact. There is an analagous sequence for every short exact sequence in the second variable.

We shall not go through the proofs of these results here, but the student should think about them. Refer back to the section on Tor for guidance. You will see this done in more detail in a course in homological algebra or you can look yourself in one of the standard references on reserve. If you study the appropriate homological algebra, you will also see why the name 'Ext' is used to describe the functor.

Note that the above facts allow us to compute Ext for every finitely generated abelian group. The additivity reduces the problem to that of cyclic groups. From the definition, $\mathrm{Ext}(Z, N) = 0$ since $\mathbf{Z}$ is free and 'is' its own presentation. The s.e.s $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ yields

$$0 \to \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, N) \to \mathrm{Hom}(\mathbf{Z}, N) \xrightarrow{n} \mathrm{Hom}(\mathbf{Z}, N) \to \mathrm{Ext}\,\mathbf{Z}/n\mathbf{Z}, N) \to 0.$$

Since $\mathrm{Hom}(Z, N) \cong N$ This shows

$$\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, N) \cong {}_nN$$
$$\mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, N) \cong N/nN.$$

It follows that (as with Tor)

$$\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}.$$

To round this out, note that $\mathrm{Hom}(\mathbf{Z}, N) = N, \mathrm{Ext}(\mathbf{Z}, N) = 0, \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) = 0, \mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$.

The lack of symmetry of Hom, as mentioned above, prevents us from proceeding in a completely symmetrical manner, as we did for Tor. It is possible to define Ext with an appropriate construction using the second argument. What is needed is a dual concept to that of a presentation, where the group N is imbedded in an appropriate kind of group Q_0. We shall not go into this in more detail here.

You may also have noted above that $\mathrm{Ext}(M, N)$ need not vanish if N is free. However, it does vanish if N has the following property: for each $x \in N$ and each positive integer n there is a $y \in N$ such that $x = ny$. A group with this property is called *divisible*, and $\mathrm{Ext}(M, N) = 0$ if N is divisible. This is easy to derive if M is finitely generated from the formula

$$\mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, N) \cong N/nN.$$

However, if M is not finitely generated, one must use transfinite induction (Zorn's Lemma). We also leave this for you discover later when you study homological algebra in greater depth. The most interesting divisible groups are $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$. Moreover, any field of characterstic zero is divisible.

Note that this means that if N is divisible, then Hom is an exact functor.

We may now apply the above results to the free presentation

$$0 \to B_n(C_*) \to C_n \to H_n(C) \to 0$$

of homology to obtain

Theorem 10.6. *Let C_* be a free chain complex, N an abelian group. Then there are natural short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Hom}(H_n(C_*), N) \to 0$$

which splits (but not naturally).

PROOF. The splitting follows by an argument analogous to the one used for homology. □

Applying this to singular cohomology, we obtain

THEOREM 10.7. *Let X be a topological space, N an abelian group. Then there are natural short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(X), N) \to H^n(X; N) \to \mathrm{Hom}(H_n(X), N) \to 0$$

which split (but not naturally).

Note also that the morphism on the right is just α which arises through evaluation of cocyles on cycles. Also, we could equally well have stated the result for the cohomology of a pair (X, A).

EXAMPLE 10.8. $H^k(S^n; N) \cong 0$ unless $k = 0, n$, in which case it is n.

EXAMPLE 10.9. For any space X, $H^k(X; \mathbf{Q})$ is a vector space over $\mathbf{Q}$ of the same dimension as the rank of $H_k(X)$.

(All the Ext terms are zero in these cases.)

EXAMPLE 10.10. $H^k(\mathbf{R}P^n; \mathbf{Z}) = 0$ in all odd degrees except if $k = n$ is odd, in which case we get $\mathbf{Z}$. It is generally $\mathbf{Z}/2\mathbf{Z}$ in even degrees $k > 0$ and of course $H^0(\mathbf{R}P^n; \mathbf{Z}) = \mathbf{Z}$.

On the other hand, $H^k(\mathbf{R}P^n, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $0 \le k \le n$.

We leave it to the student to verify these assertions and to calculate the corresponding groups for $\mathbf{C}P^n$ and $\mathbf{H}P^n$.

The above discussion of Ext works just as well for complexes which are modules over an appropriate ring K. As in the case of Tor, if K is a PID, then we get the same universal coefficient theorem, except that we need to suscript for the ring. $\mathrm{Hom}_K(M, N)$ denotes the K-module homomorphisms of $M \to N$, and Ext_K denotes the corresponding derived functor. The most interesting case is that in which K is a field. Then every K-module is free and $\mathrm{Ext}_K(M, N) = 0$ in all cases.

PROPOSITION 10.11. *Let X be a space, K a field, and N a vector space over K. Then, we have isomorphisms*

$$\alpha : H^n(X; N) \xrightarrow{\cong} \mathrm{Hom}_K(H_n(X; K), N)$$

Note that on the right hand side we have $H_n(X; K)$. The point is that the universal coefficient theorem over K refers to chain complexes defined over K, so we must apply it to $S_*(X) \otimes K$ in order to get such a chain complex.

3. Cup Products

It is unfortunately true that the algebraic invariants provided by the homology groups of a space are not adequate to tell different spaces apart. For example, it is often possible to match the homology of a space simply by taking wedges of appropriate spaces. Thus $S^{2n} \vee S^{2n-2} \vee \cdots \vee S^2$ has the same homology as $\mathbf{C}P^n$. Many similar examples abound. One trend in algebraic topology has been to supplement previously know algebraic structures with additional ones to provide greter 'resolution' in attacking geometric problems. As mentioned earlier, one advantage of *co*homology is that it may be endowed with a multiplicative structure.

Let X be a space and K a commutative ring. Common choices for K would be $K = \mathbf{Z}$ or K a field such as $\mathbf{Q}, \mathbf{R}, \mathbf{C}$, or $\mathbf{Z}/p\mathbf{Z}$ for p a prime. We shall show how to make $H^*(X; K)$ into a ring. Then it will often be the case that spaces with the same additive groups $H^k(X; K)$ in each degree may be distinguished by their multiplicative structures.

The product may be defined as follows. Let $f : S_r(X) \to K$ and $g : S_s(X) \to K$ be cochains. Define $f \cup g : S_{r+s}(X) \to K$ by

$$(f \cup g)(\sigma) = f(\sigma \circ [\mathbf{e}_0, \dots \mathbf{e}_r]) g(\sigma \circ [\mathbf{e}_r, \dots, \mathbf{e}_n]) \qquad \sigma \text{ a singular } n\text{-simplex.}$$

Using this formula, it is easy to check the following.

(i) If f, g are cocyles $f \cup g$ is also a cocyle. Also, changing f and g by coboundaries changes $f \cup g$ by a coboundary. These facts follow from the following formula which we shall leave as an exercise for the student.

$$\delta^{r+s}(f \cup g) = \delta^r f \cup g + (-1)^r f \cup \delta^s g.$$

Hence, we may define $\xi \cup \eta$ to be the class of the cocycle $f \cup g$. where f, g represent the cohomology classes $\xi \in H^r(X; K), \eta \in H^s(X; K)$ respectively.

(ii) The product so defined satisfies the appropriate distributive and associative laws. These rules in fact are already satisfied at the cochain level.

(iii) The cup product is functorial, i.e., if $\phi : X \to Y$ is a map of spaces, then $\phi^* : H^*(Y; K) \to H^*(X; K)$ preserves products, i.e.,

$$\phi^*(\xi \cup \eta) = \phi^*(xi) \cup \phi^*(\eta) \qquad \xi \in H^r(Y; K), \eta \in H^s(X; K).$$

We leave these verifications for the student. (See also the more abstract approach below.)

In general, let C^* be a cochain complex, and suppose a product structure is defined on C^* such that

(a) $C^r C^s \subseteq C^{r+s}$.

(b) The associative and distributive laws hold for the product.

(c) If $u \in C^r, v \in C^s$, then $\delta^{r+s}(uv) = (\delta^r u)v + (-1)^r u(\delta^s v)$.

Then we say that C^* is a *differential graded ring.* (It is often true that we are interested in the case where C^* is also an algebra over a field K thought of as imbedded in C^0, in which case C^* is called a differential graded algebra or 'DGA'.) Thus, the singular cochain complex of a space is a differential graded ring. In general, if C^* is a differential graded ring, then $H^*(C^*)$ becomes a *graded* ring.

There is one additional fact about the cup product that is important. It is 'commutative' in the following graded sense. If $\xi \in H^r(X;K), \eta \in H^s(X;K)$, then

$$\xi \cup \eta = (-1)^{rs} \eta \cup \xi.$$

A graded ring with this property is called a *graded commutative* ring. This rule is not so easy to derive from the formula defining the cup product for cochains, since it does not hold at the cocycle level but only holds up to a coboundary. To prove it, we shall use a more abstract approach which also gives us a better conceptual understanding of where the cup product arises. We shall show how to construct the cup product as a composite homomorphsim

$$\cup : H^*(X;K) \otimes_K H^*(X;K) \to H^*(X \times X;K) \to H^*(X;K).$$

Note that first specifying a product $H^r \times H^s \to H^{r+s}$ denoted $(u,v) \mapsto uv$ satisfies the distributive laws if and only if it is bilinear so that it induces $H^r \otimes_K H^s \to H^{r+s}$ and converses any such homomorphism induces such a product by $u \otimes v \mapsto uv$. Hence, the above is a plausible reformulation.

Secondly, the second constituent of the above homomorphism is easy to come by. Namely, let $\Delta : X \to X \times X$ and use $\Delta^* : H^*(X \times X;K) \to H^*(X)$.

The first constituent of the cup product homomorphism is a bit harder to get at. It basically comes from the Alexander–Whitney morphism. Namely, let X, Y be spaces. Then $A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ induces a morphism of cochain complexes

$$A^* : \mathrm{Hom}(S_*(X) \otimes S_*(Y), K) \to \mathrm{Hom}(S_*(X \times Y), K).$$

(Note that any other 'Eilenberg–Zilber' morphism—which would be chain homotopic to A—could be substituted here, but the Alexander–Whitney map is explicit, so it makes sense to use it.) We complement this with the morphism of cochain complexes

$$\boxtimes : \mathrm{Hom}(S_*(X);K) \otimes_K \mathrm{Hom}(S_*(Y),K) \to \mathrm{Hom}(S_*(X) \otimes S_*(Y),K)$$

defined as follows. Let $f : S_r(X) \to K, g : S_s(Y) \to K$ and define $f \boxtimes g : S_r(X) \otimes S_s(Y) \to K$ by

$$(f \boxtimes g)(\alpha \otimes \beta) = f(\alpha)g(\beta) \qquad \alpha \in S_r(X), \beta \in S_r(Y).$$

It is easy to see that the right hand side is bi-additive, so the formula defines an element of $\mathrm{Hom}(S_r(X) \otimes S_s(Y), K)$. It is also easy to see that $(f, g) \mapsto f \boxtimes g$ is bi-additive in f and g respectively. However, you will note that on the left hand side the tensor product sign has K as a subscript. That means we are treating $\mathrm{Hom}(S_*, K)$ as a module over K and then taking the tensor product as K-modules. The K- module structure is defined by $(af)(\alpha) = a(f(\alpha)$ where $a \in K, f \in \mathrm{Hom}(S_*, K)$ and $\alpha \in S_*$. That $f \boxtimes g$ is bilinear follows from

$$\begin{aligned}((af) \boxtimes g)(\alpha \otimes \beta) &= (af)(\alpha)g(\beta) = a(f(\alpha))g(\beta) \\ &= f(\alpha)(a(g(\beta)) = f(\alpha)(ag)(\beta) = (f \boxtimes (ag))(\alpha \otimes \beta).\end{aligned}$$

We shall denote the composite morphism $A^*(f \boxtimes g)$ by $f \times g$ and similarly for the induced homomorphism

$$\times : H^*(X; K) \otimes H^*(Y; K) \to H^*(X \times Y; R).$$

We now define the cup product by the rather cumbersome formula

$$f \cup g = \Delta^*(f \times g) = \Delta^*(A^*(f \boxtimes g)).$$

The student should check explicitly that this gives the formula stated above.

As we shall see, this rather indirect approach has some real advantages. For example, the naturality of each of the constituents is fairly clear, so the naturality of the cup product follows. That was already fairly from the explicit formula, but the more abstract formulation shows us how to proceed in other circumstances—as for example in cellular theory—where we might not have such an explicit formula. More to the point, it allows us to prove the graded commutative law as follows.

First, consider the effect of switching factors in the morphism

$$\boxtimes : \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \to \mathrm{Hom}(S_*(X) \otimes S_*(X), K)$$

Define $T : S_*(X) \otimes S_*(X) \to S_*(X) \otimes S_*(X)$ by $T(\sigma \otimes \tau) = (-1)^{rs}\tau \otimes \sigma$ where σ, τ have degrees r, s respectively. This is a chain morphism since

$$\begin{aligned}\partial(T(\sigma \otimes \tau)) &= (-1)^{rs}\partial(\tau \otimes \sigma) = (-1)^{rs}\partial\tau \otimes \sigma + (-1)^{rs+s}\tau \otimes \partial\sigma \\ T(\partial\sigma \otimes \tau) &= T(\partial\sigma \otimes \tau) + (-1)^r T(\sigma \otimes \partial\tau) \\ &= (-1)^{(r-1)s}\tau \otimes \partial\sigma + (-1)^{r+r(s-1)}\partial\tau \otimes \sigma.\end{aligned}$$

However, $(r-1)s = rs - s \equiv rs + s \bmod 2$ and $r + r(s-1) = rs$, so the two expressions are equal. Note that just twisting the factors without introducing a sign would not result in a chain morphism. Note also, that in degree zero, T just introduces the identity in homology

$$H_0(S_*(X) \otimes S_*(X)) \to H_0(S_*(X) \otimes S_*(X)).$$

Indeed, using a trivial example of the Künneth theorem, this is just the homomorphism $\mathbf{Z} \otimes Z \to \mathbf{Z} \otimes Z \cong \mathbf{Z}$ defined by $a \otimes b \mapsto b \otimes a \cong ba = ab$.

Similarly, $\mathrm{Hom}(S_*(X), K)$ is a cochain complex (so also a chain complex), and we have the analogous twisting morphism $T : \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \to \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K)$ defined by $f \otimes g \mapsto (-1)^{rs} g \otimes f$. It is easy to check that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) & \xrightarrow{\boxtimes} & \mathrm{Hom}(S_*(X) \otimes S_*(X), K) \\ \downarrow & & \downarrow T^* \\ \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) & \xrightarrow{\boxtimes} & \mathrm{Hom}(S_*(X) \otimes S_*(X), K) \end{array}$$

commutes, where both vertical morphisms result from twisting.

To complete the proof of the graded commutative law, it suffices to show that the two cochain morphisms

$$\begin{aligned} \Delta^* \circ A^* \circ T^*, \Delta^* A^* : \mathrm{Hom}(S_*(X) \otimes S_*(X), K) &\to \mathrm{Hom}(S_*(X \times X), K) \\ &\to \mathrm{Hom}(S_*(X), K) \end{aligned}$$

are cochain homotopic. Let X and Y be spaces, and define $t : X \times Y \to Y \times X$ by $t(x, y) = (y, x)$. Consider the diagram

$$\begin{array}{ccc} S_*(X \times Y) & \xrightarrow{A} & S_*(X) \otimes S_*(Y) \\ t_* \downarrow & & \downarrow T \\ S_*(Y \times X) & \xrightarrow{A} & S_*(Y) \otimes S_*(X) \end{array}$$

Each of the two routes from the upper left corner to the lower right corner is a natural chain map which induces the same isomorphism in homology in degree zero—the identity of $\mathbf{Z} \to \mathbf{Z}$. Hence, by the acyclic models theorem applied to the two functors $S_*(X \times Y)$ and $S_*(Y) \otimes S_*(X)$ on the category of pairs (X, Y) of spaces, it follows that the two routes yield chain homotopic morphisms. However, since

$t \circ \Delta = \Delta : X \to X \times X$, the same is true for the induced morphisms $S_*(X) \to S_*(X \times X)$, and combining this with the previous result gives us what we want.

4. Calculation of Cup Products

In this section we calculate some cohomology rings for important spaces.

Note first that if X is path connected, then $H^0(X; K)$ is naturally isomorphic to K, so we may view $1 \in K$ as an element of $H^0(X; K)$. It is not hard to see that it acts as the identity in the ring $H^*(X; K)$.

Spheres

First, note that $H^*(S^n; K)$ is easy to determine. Namely, $H^0(S^n; K) = K$, and since $H_k(S^n) = 0$ for $0 < k < n$ and $\mathbf{Z}$ for $k = n$, the universal coefficient theorem tells us that $H^k(S^n; K) = 0$ for $0 < k < n$ and $H^n(S^n; K) = K$. Let X be a generator of $H^n(S^n; K)$. By degree considerations, we must have $X \cup X = 0$. Hence, as a ring,

$$H^*(S^n; K) \cong K[X]/(X^2) \qquad \deg X = n.$$

Such a ring is called a *truncated polynomial ring.*

The 2-Torus

As above, since $H_*(T^2)$ is free, the univeral coefficient theorem yields

$$H^0(T^2; K) = K, H^1(T^2; K) = KX \oplus KY, H^2(T^2; K) = KZ$$

where we have chosen names, X, Y, Z for generators in the indicated degrees. (1 is the generator in degree 0.) Because of degree considerations

$$u \cup v = -v \cup u$$

for any elements of degree 1. In particular, if 2 is not a zero divisor in K, then

$$X \cup X = Y \cup Y = 0.$$

This will take care of $K = \mathbf{Z}$ or any field not of characteristic 2. We shall show that for the torus, these squares are zero in any case. We also have

$$X \cup Y = -Y \cup X$$

and we shall show that up to a sign these are the same as Z. The cohomology ring is an example of what is called an *exterior algebra* on the generators X, Y.

Next consider the diagram below representing T^2 as an identification space with the indicated *singular simplices.*

As usual, $H_1(T^2)$ is generated by the homology classes of the cycles e_1 and e_2. In addition, $f_1 - f_2$ is a cycle generating $H_2(T^2)$. There are several ways to see that. The simplest is to note that if we subdvide the torus further (as indicated in the above diagram), we may view it as a simplicial complex. In our previous calculation of the homology of the torus, we used exactly that simplicial decomposition. A true simplicial cycle generating $H_2(T^2)$ is obtained from $f_1 - f_2$ by sufficiently many subdivisions, and subdivision is chain homotopic to the identity.

Suppose g, h are two 1-cocycles. By the universal coefficient theorem, the cohomology classes $\overline{g}, \overline{h}$ are completely determined by $g(e_i), h(e_i)$ since in this case the Ext terms vanish and α is an isomorphism. (Remember that α amounts to evaluation at the leve of cocycles and cycles!) Similarly, the cohomology class of $g \cup h$ is completely determined by its values on $f_1 - f_2$. However,

$$\begin{aligned}(g \cup h)(f_1 - f_2) &= g(f_1 \circ [e_0, e_1])h(f_1 \circ [e_1, e_2]) - g(f_2 \circ [e_0, e_1])h(f_2 \circ [e_1, e_2) \\ &= g(e_1)h(e_2) - g(e_2)h(e_1).\end{aligned}$$

Suppose that g represents X and h represents Y, i.e., suppose $g(e_1) = 1, g(e_2) = 0, h(e_1) = 0, h(e_2) = 1$. Then

$$(g \cup h)(f_1 - f_2) = 1$$

and $g \cup h$ represents a generator of $H^2(T^2; K)$. On the other hand, suppose $g = h$ both represent X. Then the same calculations shows that $(g \cup g)(f_1 - f_2) = 0 - 0 = 0$. It follows that $X \cup X = 0$, and similarly $X \cup Y = 0$.

4.1. Cohomomology Rings of Products. Since $T^2 = S^1 \times S^1$, another approach to determining its cohomology ring is to study in general the cohomology ring of a product $X \times Y$.

We shall concentrate on the case $K = \mathbf{Z}$ *and* $H_*(X)$ and $H_*(Y)$ are free *or* K is a field. In either case, we have an isomorphism

$$H^*(X \times Y; K) \cong \mathrm{Hom}_K(H_*(X \times Y; K), K)$$

We also assume that the homology of each space is finitely generated in each degree. Under these assumptions,

$$H_*(X; K) \otimes_K H_*(Y; K) \xrightarrow{\times} H_*(X \times Y; K)$$

is an isomorphisms. For $\times$ is a monomomphism in either case by the universal coefficient theorem, and because the Tor terms vanish, it is an isomophism. Dualizing yields

$$\mathrm{Hom}_K(H_*(X; K) \otimes_K H_*(Y; K), K) \to \mathrm{Hom}_K(H_*(X \times Y; K), K) \cong H^*(X \times Y; K).$$

On the other hand it is also true that

$$\operatorname{Hom}_K(H_*(X;K),K)\otimes_K \operatorname{Hom}_K(H_*(Y;K),K) \xrightarrow{\boxtimes} \operatorname{Hom}_K(H_*(X;K)\otimes_K H_*(Y;K),K)$$

is an isomorphism. For, in each degree this amounts to explicit homomorphisms

$$\operatorname{Hom}_K(H_r,K)\otimes_K \operatorname{Hom}_K(H_s,K)\to \operatorname{Hom}_K(H_r\otimes_K H_s,K)$$

which may be shown by induction on the rank of H_r and H_s to be an isomorphism. If either H_r or H_s is of rank 1 over K, the result is obvious, and the inductive step follows using additivity of the functors.

Putting all of the above isomorphisms together, we get

$$H^*(X\times Y;K)\cong H^*(X;K)\otimes_K H^*(Y;K)$$

or more explicitly

$$H^n(X\times Y;K)\cong \bigoplus_{r+s=n} H^r(X;K)\otimes_k H^s(Y;K).$$

Hence, to describe the cup product on the left, it suffices to determine what it becomes on the right. The rule is quite simple.

$$(u_1\otimes v_1)\cup(u_2\otimes v_2)=(-1)^{\deg v_1\deg u_2}u_1\cup u_2\otimes v_1\cup u_2. \tag{45}$$

In other words, mutiply generating tensor products in the obvious way by multiplying their factors but also introduce a sign. The sign is thought of as resulting from 'moving' the second factor on the left 'past' the first factor on the right.

We shall verify this rule below, but first note that it gives the same result as above for $T^2=S^1\times S^1$. For, let $H^1(S^1;K)=Kx$ so of necessity $x\cup x=0$. Put

$$X=x\otimes 1, Y=1\otimes x\in H^1(S^1;K)\otimes_K H^0(S^1;K)\oplus H^0(S^1;K)\otimes_K H^1(S^1;K) \cong H^1(T^2;K).$$

Then

$$\begin{aligned} X\cup Y&=x\otimes 1\cup 1\otimes x=x\otimes x\\ Y\cup X&=1\otimes x\cup x\otimes 1=(-1)^{1\cdot 1}x\otimes x=-X\cup Y\\ X\cup X&=x\otimes 1\cup x\otimes 1=(x\cup x)\otimes 1=0\\ Y\cup Y&=1\otimes x\cup 1\otimes x=1\otimes(x\cup x)=0.\end{aligned}$$

The proof of formula 45 follows from the following rather involved diagram which traces the morphisms at the level of singular chains needed to define the cup product. (You should also check that dualizing gives exactly what we want by means of the various isomorphisms described above. There are some added complications if $K \neq \mathbf{Z}$, so you should first work it out for ordinary singular theory.)

This diagram commutes up to chain homotopy by the acyclic models theorem applied to the category of pairs (X, Y). As we saw previously, with the specified models, the functor $S_*(X \times Y)$ is free. Also, as before, the Künneth Theorem shows that the functor

$$S_*(X) \otimes S_*(Y) \otimes S_*(X) \otimes S_*(Y)$$

is acyclic. It is easy to check that both routes between the ends induce the same morphism in homology in degree zero.