CHAPTER 3

Quotient Spaces and Covering Spaces

1. The Quotient Topology

Let X be a topological space, and suppose $x \sim y$ denotes an equivalence relation defined on X. Denote by $\hat{X} = X/\sim$ the set of equivalence classes of the relation, and let $p : X \to \hat{X}$ be the map which associates to $x \in X$ its equivalence class. We define a topology on $\hat{X}$ by taking as open all sets $\hat{U}$ such that $p^{-1}(\hat{U})$ is open in X. (It is left to the student to check that this defines a topology.) $\hat{X}$ with this topology is called the *quotient space* of the relation.

EXAMPLE 3.1. Let $X = I$ and define $\sim$ by $0 \sim 1$ and otherwise every point is equivalent just to itself. I leave it to the student to check that $I/\sim$ is homeomorphic to S^1.

EXAMPLE 3.2. In the diagram below, the points on the top edge of the unit square are equivalent in the indicated direction to the corresponding points on the bottom edge and similarly for the right and left edges.

The resulting space is homeomorphic to the two dimensional torus $S^1 \times S^1$. To see this, map $I \times I \to S^1 \times S^1$ by $(t, s) \to (e^{2\pi i t}, e^{2\pi i s})$ where $0 \le t, s \le 1$. It is clear that points in $I \times I$ get mapped to the same point of the torus if and only they are equivalent. Hence, we get an induced one-to-one map of the quotient space onto the torus. I leave it to you to check that this map is continuous and that is inverse is continuous.

EXAMPLE 3.3. In the diagram below, the points on opposite edges are equivalent in pairs in the indicated directions. As mentioned earlier, the resulting quotient space is homeomorphic to the so-called Klein bottle. (For our purposes, we may take that quotient space to be the definition of the Klein bottle.)

An equivalence relation may be specified by giving a partition of the set into pairwise disjoint sets, which are supposed to be the equivalence classes of the relation. One way to do this is to give an onto map $f : X \to Y$ and take as equivalence classes the sets $f^{-1}(y)$ for $y \in Y$. In this case, there will be a bijection $\hat{f} : \hat{X} \to Y$, and it is not hard to see that $\hat{f}$ will be continuous. However, its inverse need not be continuous, i.e., $\hat{X}$ could have more open sets than Y. (Can you invent an example?) However, the map $\hat{f}$ will be *bicontinuous* if it is an open (similarly closed) map. In this case, we shall call the map $f : X \to Y$ a *quotient map.*

Proposition 3.4. *Let $f : X \to Y$ be an onto map and suppose X is endowed with an equivalence relation for which the equivalence classes are the sets $f^{-1}(y), y \in Y$. If f is an open (closed) map, then f is a quotient map.*

(However, the converse is not true, e.g., the map $X \to \hat{X}$ need not in general be an open map.)

Proof. If $\hat{U}$ is open (closed) in $\hat{X}$, then $p^{-1}(\hat{U})$ is open (closed) in X, and

$$\hat{f}(\hat{U}) = \hat{f}(p((p^{-1}(\hat{U})))) = f(p^{-1}(\hat{U}))$$

is open (closed) in Y. □

The above discussion is a special case of the following more general *universal mapping property* of quotient spaces.

Proposition 3.5. *Let X be a space with an equivalence relation $\sim$, and let $p : X \to \hat{X}$ be the map onto its quotient space. Given any map $f : X \to Y$ such that $x \sim y \Rightarrow f(x) = f(y)$, there exists a unique map $\hat{f} : \hat{X} \to Y$ such that $f = \hat{f} \circ p$.*

Proof. Define $\hat{f}(\hat{x}) = f(x)$. It is clear that this is defined and that $\hat{f} \circ p = f$. It is also clear that this is the only such function. To see that $\hat{f}$ is continuous, let U be open in Y. Then $f^{-1}(U)$ is open in X. But, by the definition of $\hat{f}$, $p^{-1}(\hat{f}^{-1}(U)) = f^{-1}(U)$, so $\hat{f}^{-1}(U)$ is open in $\hat{X}$. □

It makes it easier to identify a quotient space if we can relate it to a quotient map.

PROPOSITION 3.6. *Let $f : X \to Y$ be a map from from a compact space onto a Hausdorff space. Then f is a quotient map.*

(Note how this could have been used to show that the square with opposite edges identified is homeomorphic to a torus. Since the square is compact and the torus is Hausdorff, all you have to check is that the equivalence relation has equivalence classes the inverse images of points in the torus.)

PROOF. f is a closed map. For, if E is a closed subset of X, then it is compact. Hence, $f(E)$ is compact, and since Y is Hausdorff, it is closed. □

Let X be a space, and let A be a subspace. Define an equivalence relation on X by letting all points in A be equivalent and let any other point be equivalent only to itself. Denote by X/A the resulting quotient space.

EXAMPLE 3.7. Let $X = D^n$ and $A = S^{n-1}$ for $n \geq 1$. There is a map of D^n onto S^n which is one-to-one on the interior and which maps S^{n-1} to a point. (What is it? Try it first for $n = 2$.) It follows from the proposition that D^n/S^{n-1} is homeomorphic to S^n.

Quotient spaces may behave in unexpected ways. For example, the quotient space of a Hausdorff space need not be Hausdorf.

EXAMPLE 3.8. Let $X = I = [0, 1]$ and let $A = (0, 1)$. Let the equivalence classes be $\{0\}$, A, and $\{1\}$. Then $\hat{X}$ has three points: $\hat{0}, 0\hat{.}5$, and $\hat{1}$. However, the open sets are

$$\emptyset, \{0\hat{.}5\}, \{\hat{0}, 0\hat{.}5\}, \{0\hat{.}5, \hat{1}\}, \hat{X}.$$

Clearly, the problem in this example is connected to the fact that the set A is not closed.

This can't happen in certain reasonable circumstances.

PROPOSITION 3.9. *Suppose $f : X \to Y$ is a quotient map with X compact Hausdorff and f a closed map. Then Y is (compact and) Hausdorff.*

PROOF. Any singleton sets in X are closed since X is Hausdorff. Since f is closed and onto, it follows that singleton sets in Y are also closed. Choose $y, z \neq y \in Y$. Let $E = f^{-1}(y)$ and $F = f^{-1}(z)$.

Let $p \in E$. For every point $q \in F$, we can find an open neighborhood $U(q)$ of p and and open neighborhood $V(q)$ of q which don't intersect. Since F is closed, it is compact, so we can cover F with finitely many such $V(q_i), i = 1 \dots n$. Let $V_p = \cup_{i=1}^n V(q_i)$ and $U_p = \cap_{i=1}^n U(q_i)$. Then $F \subseteq V_p$ and U_p is an open neighborhood of p which is disjoint from V_p. But the collection of open sets $U_p, p \in E$ cover E, so we can pick out a finite subset such that $E \subseteq U = \cup_{j=1}^k U_{p_j}$ and U is disjoint from $V = \cap_{j=1}^k V_{p_j}$ which is still an open set containing F. We have now found disjoint open sets $U \supseteq E$ and $V \supseteq F$. Consider $f(X - U)$ and $f(X - V)$. These are closed sets in Y since f is closed. Hence, $Y - f(X - U)$ and $Y - f(X - V)$ are open sets in Y. However, $y \notin f(X - U)$ since otherwise, $y = f(w)$ with $w \notin U$, which contradicts $f^{-1}(y) \subseteq U$. Hence, $y \in Y - f(X - U)$ and similarly, $z \in Y - f(X - V)$. Thus we need only show that these two open sets in Y are disjoint. But

$$\begin{aligned}(Y - f(X - U)) \cap (Y - f(X - V)) &= Y - (f(X - U) \cup f(X - V)) \\ &= Y - f((X - U) \cup (X - V)) \\ &= Y - f(X - (U \cap V)) = Y - f(X) \\ &= \emptyset.\end{aligned}$$

□

A common application of the proposition is to the following situation.

Corollary 3.10. *Let X be compact Hausdorff, and let A be a closed subspace. Then X/A is compact Hausdorff.*

Proof. All we need to do is show that the projection $p : X \to X/A$ is closed. Let E be a closed subset of X. Then

$$p^{-1}(p(E)) = \begin{cases} E & \text{if } E \cap A = \emptyset \\ E \cup A & \text{if } E \cap A \neq \emptyset. \end{cases}$$

In either case this set is closed, so $p(E)$ is closed. □

1.1. Projective Spaces. Let $X = \mathbf{R}^{n+1} - \{0\}$. The set of lines through the origin in $\mathbf{R}^{n+1}$ is called *real projective n space* and it is denoted $\mathbf{R}P^n$. (Algebraic geometers often denote it $\mathbf{P}^n(\mathbf{R})$.) It may be visualized as a quotient space as follows. Let $X = \mathbf{R}^{n+1} - \{0\}$, and consider points equivalent if they lie on the same line, i.e., one is a non-zero multiple of the other. Then clearly $\mathbf{R}P^n$ is the quotient space and as such is endowed with a topology. It is fairly easy to see that it is Hausdorff. (Any two lines in $\mathbf{R}^n - \{0\}$ can be chosen to be the axes of open double 'cones' which don't intersect.)

Here is another simpler description. (It is helpful to concentrate on $n = 2$, i.e., the real projective plane.) Consider the inclusion $i : S^n \to \mathbf{R}^{n+1} - \{0\}$ and follow this by the projection to $\mathbf{R}P^n$. This map is clearly onto. Since S^n is compact, and $\mathbf{R}P^n$ is Hausdorff, it is a quotient map by the proposition above. (It is also a closed map by the proof of the proposition. It is also open because the image of any open set U is the same as the image of $U \cup (-U)$ which is open.) Note also that distinct points in S^n are equivalent under the induced equivalence relation if and only if they are antipodal points $(x, -x.)$ Since S^n is compact, it follows that $\mathbf{R}P^n$ is compact and Hausdorff.

Here is an even simpler description. Let

$$X = \{x \in S^n \mid x_{n+1} \geq 0\}$$

(the upper hemisphere.) Repeat the same reasoning as above to obtain a quotient map of X onto $\mathbf{R}P^n$. Note that distinct points on the bottom edge (which we may identify as S^{n-1}) are equivalent if and only if they are antipodal. Points not on the edge are equivalent only to themselves, i.e., the quotient map is one-to-one for those points.

Finally, map D^n onto $\mathbf{R}P^n$ as follows. Imbed D^n in $\mathbf{R}^{n+1}$ in the usual way in the hyperplane $x_{n+1} = 0$. Project upward onto the upper hemisphere and then map onto $\mathbf{R}P^n$ as above. Again, this yields a quotient map which is one-to-one on interior points of D^n and such that antipodal points on the boundary S^{n-1} are equivalent.

There is an interesting way to visualize $\mathbf{R}P^2$. The unit square is homeomorphic to D^2, and if we identify the edges as indicated below, we get $\mathbf{R}P^2$.

We may now do a series of 'cuttings' and 'pastings' as indicated below. (A cutting exhibits a space as the quotient of another space which is a disjoint union of appropriate spaces.)

Note the use of the Moebius band described as a square with two opposite edges identified with reversed orientation. From this point of view, the real projective plane is obtained by taking a 2-sphere, cutting a hole, and pasting a Moebius band on the edge of the hole. Of course this can't be done in $\mathbf{R}^3$ since we would have to pass the Moebius band through itself in order to get its boundary (homeomorphic to S^1) lined up properly to paste onto the edge of the hole. A Moebius band inserted in a sphere in this way is often called a cross-cap.

1.2. Group Actions and Orbit Spaces. Let G be a group and X a set. A (left) group action of G on X is a binary operation $G \times X \to X$ (denoted here $(g, x) \mapsto gx$) such that

(i) $1x = x$ for every $x \in X$.
(ii) $(gh)x = g(hx)$ for $g, h \in G$ and $x \in X$. This is a kind of associativity law.

(There is a similar definition for a right action which I leave to your imagination.)

If G acts on X, then for each $g \in G$, there is a function $L(g) : X \to X$. The rules imply that $L(1) = \mathrm{Id}_X$ and $L(gh) = L(g) \circ L(h)$. Since $L(g) \circ L(g^{-1}) = L(1) = \mathrm{Id}_X$, it follows that each $L(g)$ is in fact a bijection. Hence, this defines a function $L : G \to \mathcal{S}(X)$, the group of all bijections of X with composition of functions the group0 operation. This function is in fact a homomorphism.

This formalism may in fact be reversed. Given a homomorphism $L : G \to S(X)$, we may define a group action of G on X by $gx = L(g)(x)$.

Let G act on X. The set $Gx = \{gx \mid g \in G\}$ is called the *orbit* of x. It is in fact an equivalence class of the following relation

$$x \sim y \Leftrightarrow \exists g \in G \quad \text{such that } y = gx.$$

(That this is an equivalence relation was probably proved for you in a previous course, but if you haven't ever seen it, you should check it now.)

Suppose now that X is a topological space and G acts on X. We shall require additionally that $L(g) : X \to X$ is a continuous map for each $g \in G$. As above, it is invertible and its inverse is continuous, so it is a homeomorphism. In this case, we get a homomorphism $L : G \to \mathcal{A}(X)$, the group of all homeomorphisms of X onto itself.

Form the quotient space $X/\sim$ for the equivalence relation associated with the group action. As mentioned above, it consists of the orbits of the action. It is usually denoted X/G (although there is a reasonable argument to denote it $G\backslash X$).

EXAMPLE 3.11. Let $G = \mathbf{Z}$ and $X = \mathbf{R}$. Let $\mathbf{Z}$ act on $\mathbf{R}$ by

$$n \cdot x = n + x.$$

This defines an action. It is a little confusing to check this because the group operation in $\mathbf{Z}$ is denoted additively, with the neutral element being denoted '0' rather than '1'.

(i) $0 \cdot x = x + 0 = x$.
(ii) $(n+m)\cdot x = x+n+m = (x+m)+n = n\cdot(x+m) = n\cdot(m\cdot x)$.

The orbit of a point x is the set of all integral translates of that point. The quotient space is homeomorphic to S^1. This is easy to see by noting that the exponential map $E \to S^1$ defined earlier is in fact a quotient map with the sets $E^{-1}(z), z \in S^1$ being the orbits of this group action.

EXAMPLE 3.12. We can get a similar action by letting $\mathbf{Z}^n$ act on $\mathbf{R}^n$ by $\mathbf{n} \cdot \mathbf{x} = \mathbf{x} + \mathbf{n}$. The quotient space is the n-torus $(S^1 i)^n$.

EXAMPLE 3.13. Examples 3.11 and 3.12 are special cases of the following general construction. Assume X is a group in which the group operation is a continuous function $X \times X \to X$. Let G be a subgroup in the ordinary sense. Define an action by letting gx be the ordinary composition in X. The the orbits are the *right* cosets Gx of G in X, and the orbit space is the set of such cosets. If G is a normal subgroup, as would always be the case if the group were abelian, then X/G is just the quotient group.

EXAMPLE 3.14. Let X be any space, and consider the n-fold cartesian product $X^n = X \times X \times \cdots \times X$. Consider the symmetric group $\mathcal{S}_n$ of all permutations of $\{1, 2, \ldots, n\}$. Define an action of $\mathcal{S}_n$ on X^n by

$$\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}$$

for $\sigma \in \mathcal{S}_n$. Note the use of σ^{-1}. You should check the associativity rule here! The resulting orbit space is called a symmetric space. The case $n = 2$ is a bit easier to understand since $\sigma^{-1} = \sigma$ for the one non-trivial element of $\mathcal{S}_2$.

EXAMPLE 3.15. Let $\mathcal{S}_2 = \{1, \sigma\}$ act on S^n by $\sigma(x) = -x$. Then the orbits are pairs of antipodal points, and the quotient space is $\mathbf{R}P^n$. (For $n = 1$, the space S^1 has a group structure (multiplication of complex numbers of absolute value 1), and we can identify σ with the element -1, so $\mathcal{S}_2$ may be viewed as a subgroup of S^1. Strangely enough, this also works for S^3, but you have to know something about quaternions to understand that.)

EXAMPLE 3.16. Let $z_n = E(1/n) = e^{2\pi i/n} \in S^1$. The subgroup C_n of S^1 generated by z_n is cyclic of order n. It is not hard to see that the orbit space S^1/C_n is homeomorphic to S^1 again. Indeed a quotient map $p_n : S^1 \to S^1$ is given by $p_n(z) = z^n$. Each orbit consists of n points.

2. Covering Spaces

Let X be a path connected space. A map $p : \tilde{X} \to X$ from a path connected space $\tilde{X}$ is called a *covering map* (with $\tilde{X}$ being called a covering space) if for each point $x \in X$, there is an open neighborhood U of x such that

$$p^{-1}(U) = \bigcup_i S_i$$

is a disjoint union of open subsets of $\tilde{X}$, and for each i, $p|S_i$ is a homeomorphism of S_i onto U. An open neighborhood with this property is called *admissible*. Note that the set $p^{-1}(x)$ (called the fiber at x) is necessarily a discrete subspace of $\tilde{X}$.

EXAMPLE 3.17. Let $X = S^1, \tilde{X} = \mathbf{R}$, and $p = E$ the exponential map. More generally, let $X = (S^1)^n$ be an n-torus and let $\tilde{X} = \mathbf{R}^n$. The map E^n is a covering map.

EXAMPLE 3.18. Let $X = S^1$ (imbedded in $\mathbf{C}$), and let $\tilde{X}$ also be S^1. Let $p(z) = z^n$. This provides an n-fold covering of S^1 by itself.

EXAMPLE 3.19. Let $X = \mathbf{C}, \tilde{X} = \mathbf{C}$ and define $p : \mathbf{C} \to \mathbf{C}$ by $p(z) = z^n$. This is not a covering map. Can you see why? What happens if you delete $\{0\}$?

EXAMPLE 3.20. Let $\tilde{X} = S^n$ and $X = \mathbf{R}P^n$. Let p be the quotient map discussed earlier. This provides a two sheeted covering. For, if y is any point on S^n, we can choose an open neighborhood U_y which is disjoint from $-U_y$. Then $p(U_y)$ is an open neighborhood of $p(y)$, and $p^{-1}(p(U_y)) = U_y \cup (-U_y)$.

Let Z be a space, $z_0 \in Z$, and let $f : (Z, z_0) \to (X, x_0)$. A map $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ is said to lift f if $f = p \circ \tilde{f}$.

PROPOSITION 3.21 (Uniqueness of liftings). *Let Z be connected and let $\tilde{f}, \tilde{g} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ both lift f. Then $\tilde{f} = \tilde{g}$.*

PROOF. First, we show that the set $W = \{z \in Z \mid \tilde{f}(z) = \tilde{g}(z)\}$ is open. Let $z \in W$. Choose an admissible open neighborhood U of $f(z) \in X$, so $p^{-1}(U) = \cup_i S_i$ as above. Suppose $\tilde{f}(z) = \tilde{g}(z) \in S_i$. Let $V = \tilde{f}^{-1}(S_i) \cup \tilde{g}^{-1}(S_i)$. V is certainly an open set in Z. Moreover, for any point $z' \in V$, we have $p(\tilde{f}(z')) = f(z') = p(\tilde{g}(z'))$. Since $\tilde{f}(z'), \tilde{g}(z') \in S_i$, and p is one-to-one on S_i, it follows that $\tilde{f}(z') = \tilde{g}(z')$. Hence, $V \subseteq W$. This shows W is open.

W is certainly non-empty since it contains z_0. Hence, if we can show its complement $W' = \{z \in Z \mid \tilde{f}(z) \neq \tilde{g}(z)\}$ is also open, we can conclude it must be empty by connectedness. But, it is clear that W' is open. For, if $z \in W'$, $\tilde{f}(z)$ and $\tilde{g}(z)$ must be in disjoint components S_i and S_j of $p^{-1}(f(z))$. But then, the same is true for every point in $\tilde{f}^{-1}(S_i) \cup \tilde{g}^{-1}(S_j)$.

□

PROPOSITION 3.22 (Lifting of paths). *Let $p : \tilde{X} \to X$ be a covering map. Let $h : I \to X$ be a path starting at x_0. Let $\tilde{x}_0$ be a point in $\tilde{X}$ over x_0 Then there is a unique lifting $\tilde{h} : (I, 0) \to (\tilde{X}, \tilde{x}_0)$, i.e., such that $h = p \circ \tilde{h}$ and $\tilde{h}$ starts at $\tilde{x}_0$.*

PROOF. The uniqueness has been dealt with.

For each $x \in X$ choose an admissible open neighborhood U_x of x. Apply the Lebesgue Covering Lemma to the covering $I = \cup_x h^{-1}(U_x)$. It follows that there is a partition

$$0 = t_0 < t_1 < t_2 \cdots < t_n = 1$$

such that for each $i = 1, \ldots, n$, $[t_{i-1}, t_i] \subseteq h^{-1}(U_i)$ for some admissible open set U_i in X, i.e., $h([t_{i-1}, t_i]) \subseteq U_i$. Let h_i denote the restriction of h to $[t_{i-1}, t_i]$, Choose the component S_1 of $p^{-1}(U_1)$ containing $\tilde{x}_0$. (Recall that $p(\tilde{x}_0) = x_0 = h(0)$.) Let $\tilde{h}_1 = p_1^{-1} \circ h_1$ where $p_1 : S_1 \to U_1$ is the restriction of the covering map p. Let $x_1 = h(t_1)$ and $\tilde{x}_1 = \tilde{h}_1(t_1)$. Repeat the argument for this configuration. We get a lifting $\tilde{h}_2$ of h_2 such that $\tilde{h}_1(t_1) = \tilde{h}_2(t_1)$. Continuing in this way, we get a lifting $\tilde{h}_i$ for each h_i, and these liftings agree at the endpoints of the intervals. Putting them together yields a lifting $\tilde{h}$ for h such that $\tilde{h}(0) = \tilde{x}_0$. □

PROPOSITION 3.23 (Homotopy Lifting Lemma). *Let $p : \tilde{X} \to X$ be a covering map. Suppose $F : Z \times I \to X$ is a map such that $f = F(-, 0) : Z \to X$ can be lifted to $\tilde{X}$, i.e., there exists $\tilde{f} : Z \to \tilde{X}$ such that $f = p \circ \tilde{f}$. Then F can be lifted consistently, i.e., there exists $\tilde{F} : Z \times I \to \tilde{X}$ such that $F = p \circ \tilde{F}$ and $\tilde{f} = \tilde{F}(-, 0)$.*

Moreover, if Z is connected then $\tilde{F}$ is unique.

PROOF. The uniqueness in the connected case follows from the general uniqueness proposition proved above.

To show existence, consider first the case in which $F(Z \times I) \subseteq U$ where U is an admissible open set in X. Let $p^{-1}(U) = \cup_i S_i$ as usual, and let $p_i = p|S_i$. The sets $V_i = \tilde{f}^{-1}(S_i)$ provide a covering of Z by *disjoint* open sets. Hence, if we define

$$\tilde{F}(z, t) = {p_i}^{-1}(F(z, t)) \qquad \text{for } z \in V_i$$

we will never get a contradiction, and clearly $\tilde{F}$ lifts F.

Also, for $z \in V_i$ (unique for z), we have $p_i(\tilde{F}(z, 0)) = F(z, 0) = f(z)$, so since p_i is one-to-one, we have $\tilde{F}(z, 0) = \tilde{f}(z)$. (Note that this argument would be much simpler if Z were connected.)

Consider next the general case. For each $z \in Z, t \in I$, choose an admissible neighborhood $U_{z,t}$ of $F(z, t)$. Choose an open neighborhood

$Z_{z,t}$ of z and a closed interval $I_{z,t}$ containing t such that $F(Z_{z,t} \times I_{z,t}) \subseteq U_{z,t}$.

Fix one z. The sets $I_{z,t}$ cover I, so we may pick out a finite subset of them $I_1 = I_{t_1}, I_2 = I_{t_2}, \ldots, I_k = I_{t_k}$ which cover I. Let $V = \cap_j Z_{z,t_j}$. By the Lebesgue Covering Lemma applied to I, we can find a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ such that each $J_i = [s_{i-1}, s_i]$ is contained in some I_j. Then, each $F(V \times J_i)$ is contained in an admissible neighborhood of X, so we may apply the previous argument (with J_i replacing I). First lift $F|V \times J_1$ to $\tilde{F}_1$ so that $\tilde{F}_1(v, 0) = \tilde{f}(v)$ for $v \in V$. Next lift $F|V \times J_2$ so that $\tilde{F}_2(v, s_1) = \tilde{F}_1(v, s_1)$ for $v \in V$. Continue in this way until we have liftings $\tilde{F}_i$ for each i. Gluing these together we get a lifting $\tilde{F}_V : V \times I \to \tilde{X}$ which agrees with $\tilde{f}|V$ for $s = 0$.

We shall now show that these $\tilde{F}_V$ are consistent with one another on intersections. (So they define a map $\tilde{F} : Z \times I \to \tilde{X}$ with the right properties by the gluing lemma.) Let $v \in V \cap W$ where V and W are appropriate open sets in Z as above. Consider $\tilde{F}_V(v, -) : I \to \tilde{X}$ and $\tilde{F}_W(v, -) : I \to \tilde{X}$. These both cover $F(v, -) : I \to X$ and for $s = 0$, $\tilde{F}_V(v, 0) = \tilde{f}(v) = \tilde{F}_W(v, 0)$. Since I is connected, the uniqueness proposition implies that $\tilde{F}_V(v, s) = \tilde{F}_W(v, s)$ for all $s \in I$. However, since v was an arbitrary element of $V \cap W$, we are done. □

PROPOSITION 3.24. *Let $p : \tilde{X} \to X$ be a covering map. Let $H : I \times I \to X$ be a homotopy relative to $\dot{I}$ of paths $h, h' : I \to X$ which start and end at the same points x_0 and x_1. Let $\tilde{h}$ and $\tilde{h}'$ be liftings of h and h' respectively which start at the same point $\tilde{x}_0 \in \tilde{X}$ over x_0. Then there is a lifting $\tilde{H} : I \times I \to \tilde{X}$ which is a homotopy relative to $\dot{I}$ of $\tilde{h}$ to $\tilde{h}'$. In particular, $\tilde{h}(1) = \tilde{h}'(1)$.*

PROOF. This mimics the proof in the case of the covering $\mathbf{R} \to S^1$ which we did previously. Go back and look at it again.

Since $\tilde{H}(0, s)$ lies over $h(0) = h'(0)$ for each s, the image of $\tilde{H}(0, -)$ is contained in a discrete space (the fiber over x_0) so it is constant. A similar argument works for $\tilde{H}(1, -)$. By construction, $\tilde{H}(-, 0) = \tilde{h}$. Similarly, $\tilde{H}(-, 1)$ and $\tilde{h}'$ both lift $H(-, 1) = h'$ and they both start at $\tilde{x}_0$, so they are the same. □

COROLLARY 3.25. *Let $p : \tilde{X} \to X$ be a covering map, and choose $\tilde{x}_0$ over x_0. Then $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is a monomorphism, i.e., a one-to-one homomorphism.*

PROOF. based at $\tilde{x}_0$. If $p \circ \tilde{h}$ and $p \circ h'$ are homotopic in X relative to $\dot{I}$, then by the lifting homotopy lemma, so are $\tilde{h}$ and $\tilde{h}'$. □

3. Action of the Fundamental Group on Covering Spaces

Let $p : \tilde{X} \to X$ be a covering map, fix a point $x \in X$ and consider $\pi(X, x)$. We can define a *right* action of $\pi(X, x)$ on the fiber $p^{-1}(x)$ as follows. Let $\alpha \in \pi(X, x_0)$ and let $\tilde{x}$ be a point in $\tilde{X}$ over x. Let $h : I \to X$ be a loop at x which represents α. By the lifting lemma, we may lift h to a path $\tilde{h} : I \to \tilde{X}$ starting at $\tilde{x}$ and which by the homotopy lifting lemma is unique up to homotopy relative to $\dot{I}$. In particular, the endpoint $\tilde{h}(1)$ depends only on α and $\tilde{x}$. Define

$$\tilde{x}\alpha = \tilde{h}(1).$$

This defines a right $\pi(X, x)$ action. For, the trivial element is represented by the trivial loop which lifts to the trivial loop at $\tilde{x}$; hence, the trivial element acts trivially. Also, if β is another element of $\pi(X, x)$, represented say by a loop g, then we may lift $h * g$ by first lifting h to $\tilde{h}$ starting at $\tilde{x}$ and then lifting g to $\tilde{g}$ starting at $\tilde{h}(1)$. It follows that

$$\tilde{x}(\alpha\beta) = (\tilde{x}\alpha)\beta$$

as required for a right action.

We shall see later that this action can be extended to an action of the fundamental group on $\tilde{X}$ provided we make plausible further assumptions about $\tilde{X}$ and X.

Before proceeding, we need some more of the machinery of group actions. Before, we discussed generalities in terms of left actions, so for variation we discuss further generalities using the notation appropriate for right actions. (But, of course, with obvious notational changes it doesn't matter which side the group acts on.) So, let G act on X on the right. In this case an orbit would be denoted xG. We say the action on the set is transitive if there is only one orbit. Another way to say that is

$$\text{for each } x, y \in X, \qquad \exists g \in G \qquad \text{such that } y = xg.$$

(For example, the symmetric group $\mathcal{S}_3$ certainly acts transitively on the set $\{1, 2, 3\}$ but so does the cyclic group of order 3 generated by the cycle (123). On the other hand, the cyclic subgroup generated by the transposition (12) does not act transitively. In the latter case, the orbits are $\{1, 2\}$ and $\{3\}$.)

Proposition 3.26. *Let $p : \tilde{X} \to X$ be a covering map, and let $x \in X$. The action of $\pi(X, x)$ on $p^{-1}(x)$ is transitive.*

Proof. Let $\tilde{x}$ and $\tilde{y}$ lie over x. By assumption, since p is a covering, $\tilde{X}$ is path connected, so there is a path $\tilde{h} : I \to \tilde{X}$ starting at $\tilde{x}$ and ending at $\tilde{y}$. The projected path $p \circ \tilde{h}$ is a loop based at x, so it represents some element $\alpha \in \pi(X, x)$. From the above definition, $\tilde{y} = \tilde{x}\alpha$. □

Continuing with generalities, let G act on a set X. If $x \in X$, consider the set $G_x = \{g \in G \,|\, xg = x\}$. We leave it to the student to check that G_x is a subgroup of G. It is called the isotropy subgroup of the point x. (It is also sometimes called the stabilizer of the point x.) Let G/G_x denote the set of *right* cosets of G_x in G. (In the case of a left action, we would consider the set of left cosets instead.) Note that G/G_x isn't generally a group (unless G_x happens to be normal), but that we do have a right action of G on G/G_x. Namely, if H is any subgroup of G, the formula

$$(Hc)g = H(cg)$$

defines a right action of G on G/H. (There are some things here to be checked. First, you must know that the quantity on the right depends only on the *coset* of c, not on c. Secondly, you must check that the formula does define an action. We leave this for you to verify, but you may very well have seen it in an algebra course.)

Proposition 3.27. *Let G act on X (on the right), and let $x \in X$. Then $G_x c \mapsto xc$ defines an injection $\phi : G/G_x \to X$ with image the orbit xG. Moreover, ϕ is a map of G-sets, i.e., $\phi(\bar{c}g) = \phi(\bar{c})g$ for $\bar{c} = G_x c$ and $g \in G$.*

It follows that the index $(G : G_x)$ equals the cardinality of the orbit $|xG|$.

Note that the index and cardinality of the orbit could be transfinite cardinals, but of course to work with that you would have to be familiar with the theory of infinite cardinals. For us, the most useful case is that in which both are finite.

PROOF. First note that ϕ is well defined. For, suppose $G_x c = G_x d$. Then $cd^{-1} \in G_x$, i.e., $x(cd^{-1}) = x$. From this is follows that $xc = xd$ as required. It is clearly onto the orbit xG. To see it is one-to-one, suppose $xc = xd$. Then $x(cd^{-1}) = x$, whence $cd^{-1} \in G_x$, so $G_x c = G_x d$. We leave it as an exercise for the student to check that ϕ is a map of G sets. $\square$

Suppose x, y are in the same orbit with $y = xh$. Then

$$g \in G_y \Leftrightarrow (xh)g = xh \Leftrightarrow x(hgh^{-1}) = x \Leftrightarrow hgh^{-1} \in G_x.$$

This shows that

PROPOSITION 3.28. *Isotropy subgroups of points in the same orbit are conjugate, i.e.,* $G_{xh} = h^{-1}G_x h$.

We can make this a bit more explicit in the case of $\pi(X, x)$ acting on $p^{-1}(x)$.

COROLLARY 3.29. *Let* $p : \tilde{X} \to X$ *be a covering. The isotropy subgroup of* $\tilde{x} \in p^{-1}(x)$ *in* $\pi(X, x)$ *is* $p_*(\pi(\tilde{X}, \tilde{x}))$. *In particular,*

$$(\pi(X, x) : p_*(\pi(\tilde{X}, \tilde{x}))) = |p^{-1}(x)|.$$

Moreover, if $\tilde{y}$ *is another point in* $p^{-1}(x)$ *then*

$$p_*(\pi(\tilde{X}, \tilde{y})) = \beta^{-1} p_*(\pi(\tilde{X}, \tilde{x}))\beta$$

where $\beta \in \pi(X, x)$ *is represented by the projection of a path in* $\tilde{X}$ *from* $\tilde{x}$ *to* $\tilde{y}$.

PROOF. This is just translation. Note that the β in the second part of the Corollary is chosen so that $\tilde{y} = \tilde{x}\beta$. $\square$

As mentioned above, in the proper circumstances, the action of the fundamental group on fibers is part of an action on the covering space. Even without going that far, we can show that actions on different fibers are essentially the same. To see this, let $x, y \in X$ be two different points. Let $h : I \to X$ denote a path in X from x to y. Then, we considered before the isomorphism $\phi_h : \pi(X, x) \to \pi(X, y)$. Choose a path $\tilde{h} : I \to \tilde{X}$ over h and suppose it starts at $\tilde{x}$ over x and ends at $\tilde{y}$ over y.

PROPOSITION 3.30. *With the above notation, the following diagram commutes*

$$\begin{array}{ccc} \pi(\tilde{X}, \tilde{x}) & \xrightarrow{\phi_{\tilde{h}}} & \pi(\tilde{X}, \tilde{y}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \pi(X, x) & \xrightarrow{\phi_h} & \pi(X, y) \end{array}$$

PROOF. Let $\tilde{g}$ be a loop at $\tilde{x}$. Then

$$p \circ (\overline{\tilde{h}} * \tilde{g} * \tilde{h}) = (p \circ \overline{\tilde{h}}) * (p \circ \tilde{g}) * (p \circ \tilde{h}) = \overline{h} * (p \circ \tilde{g}) * h$$

(where $\overline{h}$ as before denotes the reverse path to h.) This says on the level of paths exactly what we want. □

COROLLARY 3.31. *If $p : \tilde{X} \to X$ is a covering, then the fibers at any two points have the same cardinality.*

PROOF. By the proposition, the isomorphism $\phi_h : \pi(X, x) \to \pi(X, y)$ carries $p_*\pi(\tilde{X}, \tilde{x})$ onto $p_*\pi(\tilde{X}, \tilde{y})$. Hence,

$$|p^{-1}(x)| = (\pi(X, x) : p_*(\pi(\tilde{X}, \tilde{x}))) = (\pi(X, y) : p_*(\pi(\tilde{X}, \tilde{y}))) = |p^{-1}(y)|.$$

□

The common number $|p^{-1}(x)|$ is called the number of sheets of the covering. For example the map $p_n : S^1 \to S^1$ defined by $p_n(z) = z^n$ provides an n-sheeted covering. Similarly, $S^1 \to \mathbf{R}P^n$ is a 2-sheeted covering for any $n \geq 1$.

EXAMPLE 3.32. The above analysis shows that

$$\pi(\mathbf{R}P^n, x_0) \cong \mathbf{Z}/2\mathbf{Z} \qquad n \geq 2$$

for any base point x_0. The argument is that

$$(\pi(\mathbf{R}P^n, x_0) : p_*(\pi(S^n, \tilde{x}_0))) = |p^{-1}(x_0)| = 2.$$

However, since S^n is simply connected, $\pi(S^n, \tilde{x}_0) = \{1\}$, so $\pi(\mathbf{R}P^n, x_0)$ has order 2.

A covering $p : \tilde{X} \to X$ is called a universal covering ($\tilde{X}$ a universal covering space) if $\tilde{X}$ is simply connected. Note that in this case

$$|p^{-1}(x)| = |\pi(X, x)|.$$

EXAMPLE 3.33. $E^n : \mathbf{R}^n \to T^n = (S^1)^n$ is a universal covering. So is $S^n \to \mathbf{R}P^n$. However, $S^1 \to S^1$ defined by $z \mapsto z^n$ is not.

4. Existence of Coverings and the Covering Group

Let X be path connected and fix $x_0 \in X$. The collection of covering maps $p : \tilde{X} \to X$ form a category. The objects are the covering maps. Given two such maps $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$, a morphism from p to p' is a map $f : \tilde{X} \to \tilde{X}'$ such that $p = p' \circ f$, i.e.,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad f \quad} & \tilde{X}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

commutes. Such morphisms are called 'maps over X'. Similarly, we can consider the category of coverings with basepoint $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, where the morphisms are maps over X preserving basepoints.

In the basepoint preserving category, the uniqueness lemma assures us that if there is a map $f : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}', \tilde{x}'_0)$ over X, it is unique. In this case, we may say the first covering with base point *dominates* the other. Domination behaves like a partial order on the collection of coverings in the sense that it is reflexive and transitive—which you should prove—but two coverings can dominate each other without being the same. On the other hand, in the first category (ignoring base points), there may be many maps between objects. In particular, we may consider the collection of *homeomorphisms* $f : \tilde{X} \to \tilde{X}$ over X. This set forms a group under composition. For it is clear that it is closed under composition, that $\mathrm{Id}_{\tilde{X}}$ is in it, and that the inververse of a map over X is a map over X. This group is called the covering group of the covering and denoted $\mathrm{Cov}_X(\tilde{X})$.

PROPOSITION 3.34. *Let $p : \tilde{X} \to X$ be a covering map, and let $x \in X$. The actions of $\mathrm{Cov}_X(\tilde{X})$ and $\pi(X, x)$ on the fiber $p^{-1}(x)$ are consistent, i.e.,*

$$f(\tilde{x}\alpha) = f(\tilde{x})\alpha$$

for $f \in \mathrm{Cov}_X(\tilde{X}), \alpha \in \pi(X, x), \tilde{x} \in p^{-1}(x)$.

PROOF. Let $[h] = \alpha$ where h is a loop in X at x. Let $\tilde{h}$ be the unique path over h starting at $\tilde{x}$. Since $p = p \circ f$, $f \circ \tilde{h}$ is the unique path over h starting at $f(\tilde{x})$. We have

$$f(\tilde{x})\alpha = (f \circ \tilde{h})(1) = f(\tilde{h}(1)) = f(\tilde{x}\alpha).$$

□

In this rest of this section we want to explore further the relation between these actions. We shall see that in certain circumstances, they are basically the same.

The first question we shall investigate is the existence of maps between covering spaces. We state the relevant lemma in somewhat broader generality.

PROPOSITION 3.35 (Existence of liftings). *Let* $p : \tilde{X} \to X$ *be a covering, let* $x_0 \in X$ *and let* $\tilde{x}_0$ *lie over* x_0. *Let* $f : (Z, z_0) \to (X, x_0)$ *be a basepoint preserving map of a connected, locally path connected space* Z *into* X. *Then* f *can be lifted to a map* $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ *if and only if*

$$f_*(\pi(Z, z_0)) \subseteq p_*(\pi(\tilde{X}, \tilde{x}_0)).$$

Recall: A space Z is locally path connected if given any point $z \in Z$ and a neighborhood V of z, there is a smaller open neighborhood $W \subseteq V$ of z which is path connected. A space can be path connected without being locally path connected. (Look at the homework problems. A space introduced in another context is an example. Which one is it?) However, if Z is locally path connected, then it is connected if and only if it is path connected.

The most important application of the above Proposition is to the case in which Z is simply connected because then the condition is certainly verified. In particular, suppose X is locally path connected, so in fact any covering space is locally path connected. Suppose in addition that there is a universal covering $p : \tilde{X} \to X$. According to this proposition, this maps over $\tilde{X}$ to any other covering, and in the base point preserving category, such a map is unique. Hence, under these hypotheses, a univeral covering space with base point is a 'largest' object in the sense that it dominates every other object.

PROOF. If such a map $\tilde{f}$ exists, it follows from $f_* = p_* \circ \tilde{f}_*$ that the required relation holds between the two images in $\pi(X, x_0)$.

Define $\tilde{f} : (Z, z_0) \to (\tilde{X}, x_0)$ as follows. Let $z \in Z$. Choose a path h in Z from z_0 to z. (Z is path connected.) Lift $f \circ h$ to a path $\tilde{g} : I \to \tilde{X}$ such that $\tilde{g}(0) = \tilde{x}_0$, and let

$$\tilde{f}(z) = \tilde{g}(1).$$

If this function is well defined and continuous, it satisfies the desired conctions. For, if $z = z_0$, we may choose the trivial path at z_0, so $\tilde{f}(z_0) = \tilde{x}_0$. Also,

$$p(\tilde{f}(z)) = p(\tilde{g}(1)) = f(h(1)) = f(z)$$

so $\tilde{f}$ covers f.

It remains to show that the above definition is that of a continuous map. First we note that it is well defined. For, suppose $h' : I \to Z$ is another path from z_0 to z. Then $h' * \overline{h}$ is a loop in Z at z_0.

Thus,

$$(f \circ h') * (\overline{f \circ h}) = f \circ (h' * \overline{h}) \sim_{\dot{I}} p \circ \tilde{j}$$

for some loop $\tilde{j} : I \to \tilde{X}$ at $\tilde{x}_0$. (That is a translation of the statement that $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$.). Hence,

$$f \circ h' \sim_{\dot{I}} (p \circ \tilde{j}) * (f \circ h) = (p \circ \tilde{j}) * (p \circ \tilde{g}) = p \circ (\tilde{j} * \tilde{g}).$$

Hence, by the homotopy lifting lemma, any lifting $\tilde{g}'$ of $f \circ h'$ starting at $\tilde{x}_0$ must end in the same place as $\tilde{g}$, i.e., $\tilde{g}'(1) = \tilde{g}(1)$. Thus the function $\tilde{f}$ is well defined. To prove $\tilde{f}$ is continuous, we need to use the hypothesis that Z is locally path connected. Let $\tilde{U}$ be an open set in $\tilde{X}$. We shall show that $\tilde{f}^{-1}(\tilde{U})$ is open. Let $z \in \tilde{f}^{-1}(\tilde{U})$. Choose an admissible open neighborhood W of $f(z)$ and let S be the component of $p^{-1}(W)$ containing $\tilde{f}(z)$. Then since $p|S$ is a homemorphism, the set $p(S \cap \tilde{U})$ is an open neighborhood of $f(z)$. Hence $f^{-1}(p(S \cap \tilde{U}))$ is an open neighborhood of z and we may choose a path connected open neighborhood V of z contained in it. Let $v \in V$. Choose a path h from z_0 to z, a path k in V from z to v and let $h' = h * k$.

Lift $f \circ h'$ as follows. First lift $f \circ h$ to $\tilde{g}$ as before so $\tilde{f}(z)$ is the endpoint of $\tilde{g}$. Inside $S \cap \tilde{U}$ lift $f \circ k$ which is a path in $p(S \cap \tilde{U})$ by composing with $(p|S)^{-1}$ (which is a homeomorphism). The result $\tilde{l}$ will be a path in $S \cap \tilde{U}$ starting at the endpoint of $\tilde{g}$. Hence, the path $\tilde{g} * \tilde{l}$ lifts $f \circ (h * k) = (f \circ h) * (f \circ k)$. It follows that $\tilde{f}(v)$ which is the endpoint of $\tilde{g} * \tilde{l}$ lies in $S \cap \tilde{U}$. Hence, $V \subseteq \tilde{f}^{-1}(\tilde{U})$, which shows that every point in $\tilde{f}^{-1}(\tilde{U})$ has an open neighborhood also contained in that set. Hence, $\tilde{f}^{-1}(\tilde{U})$ is open as required. $\square$

The proposition gives us a better way to understand the category of coverings with base point $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of X. There is a (unique) map in this category $f : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}'_0)$, i.e., the first covering dominates the second, if and only if $p_*(\pi(\tilde{X}, \tilde{x}_0)) \subseteq p'_*(\pi(\tilde{Y}, \tilde{y}_0))$. If the images of the two fundamental groups are equal, each dominates the other, which is to say that they are isomorphic objects in the category of coverings with base points. Thus, there is a one-to-one correspondence between isomorphism classes of coverings with basepoints and a certain collection of subgroups of $\pi(X, x_0)$. (We shall see later that if there is a universal covering space, then every subgroup arises in this way.) Moreover, the ordering of such isomorphism classes under domination is reflected in the ordering of subgroups under inclusion.

The category of coverings (ignoring basepoints) is a bit more complicated. If $p : \tilde{X} \to X$ is a covering, then by a previous propostion (—find it—) the subgroups $p_*(\pi(\tilde{X}, \tilde{x}_0))$ for $\tilde{x}_0 \in p^{-1}(x_0)$ form a complete set of conjugate subgroups of $\pi(X, x_0)$, i.e., a *conjugacy class*. If $p : \tilde{X}' \to X$ is another covering which yields the same conjugacy class, then each $p_*(\pi(\tilde{X}', \tilde{x}'_0))$ (for $\tilde{x}'_0 \in p'^{-1}(x_0)$) is conjugate to some $p_*(\pi(\tilde{X}, \tilde{x}_0))$, i.e.,

$$p_*(\pi(\tilde{X}', \tilde{x}'_0)) = \beta^{-1} p_*(\pi(\tilde{X}, \tilde{x}_0))\beta = p_*(\pi(\tilde{X}, \tilde{x}_0\beta)).$$

It follows from the existence lemma above that there is an isomorphism $f : \tilde{X}' \to \tilde{X}$ over X (carrying $\tilde{x}'_0$ to $\tilde{x}_0\beta$). Thus, the isomorphism classes of coverings over X are in one-to-one correspondence with a certain collection of conjugacy classes of subgroups of $\pi(X, x_0)$. (Again, we shall see later that every such conjugacy class arises from a covering if X has a universal covering space.)

Return now to a single covering $p : \tilde{X} \to X$, and consider the covering group $\mathrm{Cov}_X(\tilde{X})$. Fix a base point $x \in X$ and $\tilde{x} \in \tilde{X}$ over x. Let $f \in \mathrm{Cov}_X(\tilde{X})$. Then by the uniqueness lemma, f is completely determined by the image $f(x)$. Also,

$$f(\tilde{x}) = \tilde{x}\gamma$$

for some $\gamma \in \Pi = \pi(X, x)$. Of course, γ is not unique. However,

$$\tilde{x}\gamma = \tilde{x}\delta \Leftrightarrow \tilde{x} = \tilde{x}\gamma\delta^{-1} \Leftrightarrow \gamma\delta^{-1} \in \Pi_{\tilde{x}} = p_*(\pi(\tilde{X}, \tilde{x})).$$

Hence, each f uniquely determines a coset $(\Pi_{\tilde{x}})\gamma$. Clearly, not every coset in $\Pi/\Pi_{\tilde{x}}$ need arise in this way. Indeed, there is a map $f : \tilde{X} \to \tilde{X}$ over X carrying $\tilde{x}$ to $\tilde{x}\gamma$ if and only if $p_*(\pi(\tilde{X}, \tilde{x})) = \Pi_{\tilde{x}} \subseteq p_*(\pi(\tilde{X}, \tilde{x}\gamma)) = \Pi_{\tilde{x}\gamma} = \gamma^{-1}\Pi_{\tilde{x}}\gamma$. However, this is the same as saying

$$\gamma\Pi_{\tilde{x}}\gamma^{-1} \subseteq \Pi_{\tilde{x}}.$$

The set of all γ with this property is called the normalizer of $\Pi_{\tilde{x}}$ in Π and is denoted $N_\Pi(\Pi_{\tilde{x}})$. It is easy to check that it is a subgroup of Π. In fact, it is the largest subgroup of Π which $\Pi_{\tilde{x}}$ is normal in. We have now almost proved the following proposition.

PROPOSITION 3.36. *Let X be a locally path connected, connected space, and let $p : \tilde{X} \to X$ be a covering. Let $x \in X$ and let $p(\tilde{x}) = x$. Then*

$$\mathrm{Cov}_X(\tilde{X}) \cong N_{\pi(X,x)}(p_*(\pi(\tilde{X},\tilde{x})))/p_*(\pi(\tilde{X},\tilde{x})).$$

PROOF. We need only prove that the map $f \mapsto (\Pi_{\tilde{x}})\gamma$ defined by $f(\tilde{x}) = \tilde{x}\gamma$ is a homomorphism. Let f' be another element of $\mathrm{Cov}_X(\tilde{X})$. Then

$$f'(f(\tilde{x})) = f'(\tilde{x}\gamma) = f'(\tilde{x})\gamma = (\tilde{x}\gamma')\gamma = \tilde{x}(\gamma'\gamma).$$

Hence $f' \circ f \mapsto (\Pi_{\tilde{x}})\gamma'\gamma$ as required. □

Note that much of this discussion simplifies in case $\pi(X, x)$ is abelian. For, in that case every subgroup is normal, and a conjugacy class consists of a single subgroup. Each covering corresponds to a single subgroup of the fundamental group and the covering group is the quotient group.

4.1. Existence of Covering Spaces. We now address the question of how to construct covering spaces in the first place. One way to do this is to start with a covering $p : \tilde{X} \to X$ and to try to construct coverings it dominates (after choice of a base point.) Suppose in particular that the covering has the property that $\Pi_{\tilde{x}} = p_*(\pi(\tilde{X}, \tilde{x}))$ is a *normal* subgroup of $\Pi = \pi(X, x)$. We call such a covering a *regular* or *normal* covering. Note in particular that any univeral covering is necessarily normal.

PROPOSITION 3.37. *Assume X is connected and locally path connected. Let $p : \tilde{X} \to X$ be a normal (regular) covering, and let $p(\tilde{x}) = x$. Then*

$$\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)/p_*(\pi(\tilde{X}, \tilde{x})).$$

For a universal covering, we have

$$\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x).$$

EXAMPLE 3.38. $E^n : \mathbf{R}^n \to T^n$ is a normal covering, so

$$\mathrm{Cov}_{T^n}(\mathbf{R}^n, \tilde{x}) \cong \pi(T^n, x) \cong \mathbf{Z}^n.$$

In this case the action is fairly easy to describe. Take $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbf{Z}^n$. Then $\mathbf{j} \cdot \mathbf{x} = \mathbf{j} + \mathbf{x}$. (If you look back at the proof that $\pi(S^1) \cong \mathbf{Z}$, which was done by liftings, you will see that we verified this in essence for $n = 1$. You should check that this reasoning can be carried through for $n > 1$.)

$p : S^n \to \mathbf{R}P^n$ for $n > 1$ is a universal covering, so $\mathrm{Cov}_{\mathbf{R}P^n}(S^n) \cong \mathbf{Z}/2\mathbf{Z}$. It is clear that the antipodal map is a nontrivial element of the covering group, so the covering group consists of the identity and the antipodal map.

PROPOSITION 3.39. *Suppose $p : \tilde{X} \to X$ is a covering where X is locally path connected and connected. Then the action of $G = \mathrm{Cov}_X(\tilde{X})$ on $\tilde{X}$ has the following property: for any $\tilde{x} \in \tilde{X}$ there is an open neigborhood $\tilde{U}$ of $\tilde{x}$ such that $g(U) \cap U = \emptyset$ for every $g \in G$.*

An action of a group on a space by continuous maps is called *properly discontinuous* if the above condition is met. Note that any two translates $g(U)$ and $h(U)$ of U by elements of G are disjoint. (Proof?)

PROOF. Let $x = p(\tilde{x})$, choose an admissible open connected neighborhood U of x, and let $\tilde{U}$ be the component of $p^{-1}(U)$ containing $\tilde{x}$. $g(\tilde{U})$ must be a connected component of $p^{-1}(U)$ and since $g(\tilde{x}) \neq \tilde{x}$, it can't be $\tilde{U}$. □

We would now like to be able to reverse this reasoning.

PROPOSITION 3.40. *Let G be a group of continuous maps of a connected, locally path connected space $\tilde{X}$, and suppose the action is properly discontinuous. Then the quotient map $p : \tilde{X} \to X = \tilde{X}/G$ is a regular covering, and $\mathrm{Cov}_X(\tilde{X}) = G$.*

PROOF. First note that p is an open map. For, let $\tilde{U}$ be an open set in $\tilde{X}$. Then, since each $g \in G$ is in fact a homeomorphism (which follows from the defintion of the action of a group on a space), it follows that each $g(\tilde{U})$ is open. Hence,

$$p^{-1}(p(\tilde{U})) = \bigcup_{g \in G} g(\tilde{U})$$

is open. So, by the definition of the topology in the quotient space, $p(\tilde{U})$ is open.

It now follows that p is a covering. For, given $x \in X$, pick $\tilde{x}$ lying over x and let $\tilde{U}$ be an open neighborhood of $\tilde{x}$ such that the open sets $g(\tilde{U})$ for $g \in G$ are all disjoint. Since p is an open map, $U = p(\tilde{U})$ is an open neighborhood of x and $p^{-1}(\tilde{U}) = \cup_g g(\tilde{U})$ is a disjoint union of open sets. Moreover, by the definition of the set $\tilde{X}/G$, $p|\tilde{U}$ is certainly one-to-one and onto, and since it is continuous and open, it is a homeomorphism. Finally, since $p \circ g = p$, the same is true for $p|g(\tilde{U}) \to U$ for any $g \in G$.

To show that $G = \mathrm{Cov}_X(\tilde{X})$, first note that G is a group of covering maps over X, so $G \subseteq \mathrm{Cov}_X(\tilde{X})$. To see that they are equal, consider $f(\tilde{x}) \in p^{-1}(x)$ for $f \in \mathrm{Cov}_X(\tilde{X})$. The fiber is just an orbit under the action of G, so

$$f(\tilde{x}) = g(\tilde{x})$$

for an appropriate $g \in G$. By the uniqueness lemma, $f = g$.

Finally, we show that the covering is normal (regular). Let $\tilde{x}, \tilde{y} = \tilde{x}\alpha$ be arbitrary points in $p^{-1}(x)$. Then, since as above, $\tilde{y} = g(\tilde{x})$ for some covering map g, it follows that $p_*(\pi(\tilde{X}, \tilde{x})) \subseteq p_*(\pi(\tilde{X}, \tilde{y})) = \alpha^{-1} p_*(\pi(\tilde{X}, \tilde{x}))\alpha$. Since α is arbitrary, it is easy to see that $p_*(\pi(\tilde{X}, \tilde{x}))$ is normal as required. □

Note that it is fairly clear that the action of $\mathrm{Cov}_X(\tilde{X})$ on $\tilde{X}$ is properly discontinous for any covering $\tilde{X} \to X$. It is natural to ask then if the quotient space for this action is X again. In fact, this will happen only in the case that the covering is regular. We leave it to the student to check that this is true.

Suppose now that $p : \tilde{X} \to X$ is a universal covering where X is connected and locally path connected. Let H be any subgroup of $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$. H certainly acts properly discontinuously on $\tilde{X}$, so we may let $\tilde{X}' = \tilde{X}/H$. Then, the quotient map $q : \tilde{X} \to \tilde{X}'$ is a regular covering with covering group H. (See the Exercises.) Define $p' : \tilde{X}' \to X$ by $r(\tilde{x}') = p(\tilde{x})$ where $\tilde{x} \in q^{-1}(\tilde{x}')$. Since all such $\tilde{x}$ are related by elements of H, they are in a single fiber for p, so they project to the same element $p(\tilde{x})$. Thus, p' is well defined. We leave it to the student to show that p' is continuous and a covering map.

Lemma 3.41. *With the above notation, $q : \tilde{X} \to \tilde{X}'$ is consistent with the action of $\pi(X, x)$ on fibers, i.e.,*

$$q(\tilde{x}\alpha) = q(\tilde{x})\alpha$$

for $\tilde{x} \in \tilde{X}$ and $\alpha \in \pi(X, p(\tilde{x}))$.

Proof. Use the fact that $q(\tilde{x})$ is the orbit $H\tilde{x}$ and that the actions of $\mathrm{Cov}_X(\tilde{X})$ and $\pi(X, x)$ on $p^{-1}(x)$ (where $x = p(\tilde{x})$) are consistent. □

Proposition 3.42. *Let X be connected and locally path connected and suppose X has a universal covering $p : \tilde{X} \to X$. Let $x \in X$. Then every subgroup H' of $\pi(X, x)$ is of the form $p'_*(\pi(\tilde{X}', \tilde{x}'))$ for some covering $p' : \tilde{X}' \to X$.*

What this proposition tells us, together with what was proved before, is that there is a one-to-one correspondence between the collection of isomorphism classes of coverings with base points and the collection of subgroups of the fundamental group of X. Similarly, there is a one-to-one correspondence between the collection of isomorphism classes of coverings (ignoring base points) and the collection of conjugacy classes of subgroups of the fundamental group.

PROOF. An isomorphism $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$ may be specified as follows. Let $p(\tilde{x}) = x$. Then

$$g \leftrightarrow \alpha \qquad \text{if and only} \qquad g(\tilde{x}) = \tilde{x}\alpha.$$

Let H correspond to H' under this isomorphism, and consider the covering $p' : \tilde{X}' = \tilde{X}/H \to X$ as above. We have

$$\begin{aligned} \tilde{x}'\alpha = \tilde{x}' &\Leftrightarrow q(\tilde{x}\alpha) = q(\tilde{x})\alpha = q(tx) \\ &\Leftrightarrow \tilde{x}\alpha = h(\tilde{x}) \quad \text{for some } h \in H. \end{aligned}$$

However, this just says that α fixes $\tilde{x}'$ if and only if it corresponds under the isomorphism to an element of H, i.e., if and only if it is an element of H'. Hence,

$$H' = p'_*(\pi(\tilde{X}', \tilde{x}'))$$

as claimed. □

EXAMPLE 3.43. Consider the universal covering $E^n : \mathbf{R}^n \to T^n$. Fix a point $x \in T^n$. The fundamental group $\pi(T^n, x) \cong \mathbf{Z}^n$ as mentioned earlier. Also, $\mathrm{Cov}_{T^n}(\mathbf{R}^n)$ consists of all translations of $\mathbf{R}^n$ by vectors $\mathbf{k} = (k_1, \dots, k_n)$ with integral components. Hence, we can determine all coverings of T^n by describing all subgroups of $\mathbf{Z}^n$.

First consider the case $n = 1$. Then every non-trivial subgroup of $\mathbf{Z}$ is of the form $H = m\mathbf{Z}$ for some positive integer m. The corresponding covering space $\tilde{X}' = \mathbf{R}/H$ is homemorphic to S^1 again, but where instead of identifying points in $\mathbf{R}$ which are 1 unit apart, we identify points which are m units apart.

The case $n > 1$ is similar but the algebra is more complicated. An example for $n = 2$ is indicated diagramatically below.

4.2. Existence of Universal Covering Spaces.

THEOREM 3.44. *Let X be a connected, locally path connected space. Then, X has a universal covering space if and only if it satisfies the following property: for each point $x \in X$ there is an open neighborhood U of x such that every loop in U at x is homotopic in X (relative to $\dot{I}$) to the constant loop x.*

This property has the confusing name 'semi-locally simply connected.'

You can find the proof of this theorem in Massey and elsewhere.

5. Covering Groups

Let X be a connected, locally connected space, and suppose it has a universal covering space $\tilde{X}$. Suppose in addition that X has the structure of a *topological group*, i.e., it is a group in which the group operation and inverse map are continuous. Then, it is possible to show that $\tilde{X}$ may also be endowed with the structure of a topological group such that the covering map $p : \tilde{X} \to X$ is a group epimorphism. (See Massey—which has some exercises with hints to prove this—or one of the other references on covering spaces.) In this case, it is easy to see that we can identify the kernel K of the homomorphism p with $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$. For, if $\tilde{k} \in K$, then the map $f_{\tilde{k}}$ defined by

$$f_{\tilde{k}}(\tilde{x}) = \tilde{K}\tilde{x}$$

is easily seen to be a covering map. Since $f_{\tilde{k}}(\tilde{1}) - \tilde{k}$, $\tilde{k} \mapsto f_{\tilde{k}}$ is a monomorphism. Also, by group theory, the fiber over any point $p^{-1}(x)$ is just the coset $\tilde{x}K = K\tilde{x}$ for any $\tilde{x} \in p^{-1}(x)$. If $f \in \mathrm{Cov}_X(\tilde{X})$, then there is a $\tilde{k} \in K$ such that $f(\tilde{x}) = \tilde{k}\tilde{x} = f_{\tilde{k}}(\tilde{x})$ so $f = f_{\tilde{k}}$. Hence, $\tilde{k} \mapsto f_{\tilde{k}}$ is an isomorphism.

EXAMPLE 3.45 (Torii). Consider $E^n : \mathbf{R}^n \to T^n$. The kernel is $\mathbf{Z}^n$ and this is the covering group as mentioned before.

EXAMPLE 3.46 (The rotation group). First consider the group $Gl(n\mathbf{R})$ of all $n \times n$ invertible matrices. Viewed as a subset of $\mathbf{R}^{n^2}$ it becomes a topological group. It is not connected, but if we take the collection of all invertible matrices with positive determinant, then this is path connected. (See the Exercises.) This collection of matrices is the path component containing the identity and it is a normal subgroup S of $Gl(n, \mathbf{R})$. There is one other component, namely RS where R is the matrix with -1 in the $1, 1$ position and is otherwise the same as the identity matrix. (R represents a reflection. Any other reflection would do as well.) These facts are tied up with the idea of orientation in $\mathbf{R}^n$. Namely, all possible bases are divided into two classes. Those determined by coordinate transformation matrices with positive determinant and those determined by coordinate transformations with negative determinant. These are called the orientation classes of the bases, and a reflection switches orientation.

Consider in particular the case $n = 3$ and consider the subgroup $O(3)$ of $Gl(3, \mathbf{R})$ consisting of all orthogonal 3×3 matrices. Since the determinant of an orthogonal matrix is ± 1, $O(3) \cap S$ consists of all 3×3 orthogonal matrices of determinant 1, and is denoted $SO(3)$. $SO(3)$ is also path connected. To see this among other things, we shall construct a surjective map $p : S^3 \to SO(3)$ with the property that the inverse image of every point in $SO(3)$ is a pair of antipodal points in S^3. It follows from the existence of such a map that $SO(3)$ is homeomorphic to $\mathbf{R}P^3$ and that $p : S^3 \to SO(3)$ is a universal covering. By what we noted above, S^3 must have the structure of a topological group, and $\pi(SO(3), I)$ may be identified with the kernel of the projection p. S^3 with this group structure is denoted $Spin(3)$, and the kernel is cyclic of order two. (The only S^n which can be made into topological groups are S^1 and S^3.)

To define the map p, first consider D^3. We shall define a surjection $p : D^3 \to SO(3)$ which is one-to-one on the interior and maps antipodal points on the boundary S^2 to the same point. (Then as before, we may produce from this a quotient map from S^3 to $SO(3)$ which sends antipodal points to the same point by using the usual relation between the upper 'hemisphere' of S^3 and D^3.) $p : D^3 \to SO(3)$ is defined as follows. Let $p(0) = I$. For $x \neq 0 \in D^3$, let $p(x)$ be the rotation of $\mathbf{R}^3$ about the axis x and through the angle $|x|\pi$ (using the right hand rule to determine the direction to rotate). The matrix of the rotation $p(x)$ with respect to an appropriate coordinate system has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta = |x|\pi$, so it is orthogonal with determinant 1. The only ambiguity in describing $p(x)$ is that rotation about the axes x and $-x$ through angle π are the same, which is to say that the mapping is one-to-one except for points with $|x| = 1$ where it maps each pair of antipodal points to the one point.

We leave it to the student to verify that p is a continous map.

The only thing remaining is to show that p is onto. We do this by showing that every non-trivial element of $SO(3)$ is a rotation. First note that any orthogonal matrix 3×3 matrix A has at least one real eigenvalue since its characteristic equation is a cubic. The absolute value of that eigenvalue must be 1, Since the product of the complex eigenvalues of A is $\det A = 1$, it is not hard to see that at least one eigenvalue must be 1. This says the correpsonding eigenvector v is in fact fixed by A. Change to an orthonormal bais with v the first basis vector. With respect to this basis, A has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is a 2×2 orthogonal matrix of determinant 1. However, it is not hard to see that any such matrix must be of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

i.e., it is a 2×2 rotation matrix. This in fact shows that A is the rotation about the axis v through angle θ, and we may certainly assume $0 \le \theta \le \pi$.