MATH 795 Algebraic Topology – Problem Set One – SOLUTIONS Spring 2010

1. Prove that the product of two Hausdorff topological spaces is Hausdorff.

Solution: Let (x, y) and (x', y') be two distinct points in $X \times Y$. Then *either* $x \neq x'$ or $y \neq y'$ (or both). Suppose $x \neq x'$. Then since X is Hausdorff, there are disjoint open sets U and V in X with $x \in U$ and $x' \in V$. Then $U \times Y$ and $V \times Y$ are disjoint open sets in $X \times Y$ with $(x, y) \in U \times Y$ and $(x', y') \in V \times Y$. If $x = x'$ and $y \neq y'$ the argument is similar.

2. Prove that a retract of a Hausdorff space is closed.

Solution: Let $i : A \to X$ be inclusion of a subset, and let $r : X \to A$ be a retraction. This means $A = \{x \in X | x = r(x)\}$. We'll show that $X - A$ is open. Let $x \in X - A$ so $x \neq r(x)$. Since X is Hausdorff, there are open disjoint sets U and V with $x \in U$ and $r(x) \in V$. Now, $V \cap A$ is open in A. So $r^{-1}(V \cap A)$ is open in X. Since $x \in U$ and $x \in r^{-1}(V \cap A)$ and the intersection of two open sets is open, we have that $\tilde{U} = U \cap r^{-1}(V \cap A)$ is an open set in X containing x. Now notice that $\tilde{U} \cap A =$ so $X - A$ is open, i.e. A is closed.

Second solution: Let x be in the closure of A, let $\{x_i\}$ be a sequence of points in A, converging to x. We need to show $x \in A$. Apply r: then $\{r(x_i)\}$ is a sequence of points converging to $r(x)$. Now $r(x_i) = x_i$ for all i, so $\{x_i\}$ is converging to $r(x)$ also. Since X is Hausdorff, the limit of a sequence is unique, so $x = r(x)$, thus $x \in A$. (**N.B.** For an arbitrary topological space, an element in the closure of a set may not be the limit of a *sequence* of points in the set. This proof can be made to work, but the concept of sequence has to be generalized to the concept of a 'net'.)

3. Let T be a torus (i.e. $T = S^1 \times S^1$), and let x_0 be a point in T. Show that $T - \{x_0\}$ has a 'figure eight' as a deformation retract.

Solution: Take a square and identify opposite edges, i.e. think of the torus as the quotient space of $I \times I$ with the identification $(t, 0) \sim (t, 1)$ and $(0, s) \sim (1, s)$. Now let x_0 be the point on the torus which is the image of the center of the square $(1/2, 1/2)$. The boundary of the square is a deformation retract of $I \times I - \{(1/2, 1/2)\}$. So the image of the boundary under the quotient map is a deformation retract of the whole torus minus a point. The image of the boundary under the identification is a figure eight.

4. Let X be a simply connected topological space, and let x and y be two distinct points in X. Show that there is a unique path class in X with initial point x_0 and terminal point y_0.

Solution: Let f and g be two paths from x_0 to y_0. We need to show that f is homotopic to g relative to $\dot{I}$. Consider the loop $f \star \bar{g}$ which is the identity in $\pi_1(X, x_0)$ since X is simply connected. So there is a homotopy $H : I \times I \to X$ with $H(t, 0) = (f \star g)(t)$ and $H(t, 1) = x_0$ and $H(0, s) = H(1, s) = x_0$. Now consider the path $h(s) = H(1/2, s)$ which starts at y_0 and ends at x_0. If you picture the square on which H is defined, this is given by restricting H to the vertical line down the middle. Looking at the left half of the square, you can produce a homotopy, relative to $\dot{I}$ between f and $\bar{h}$. The right half of the square gives a homotopy, relative to $\dot{I}$ between $\bar{h}$ and g. By transitivity of the homotopy relation, f is homotopic, relative $\dot{I}$, to g.

Second Solution: There is a very simple algebraic proof of this fact, if one works in what is called the 'fundamental groupoid' of X. A groupoid is a set with binary operation which is associative, and there are identity elements, and every element has an inverse. The catch is, not every pair of elements can be multiplied. Every element is said to have a *source* and a *target*, and α can be multiplied by β only when the target of α equals the source of β. The homotopy equivalence classes of paths, relative to $\dot{I}$ form a groupoid, where the source of a path is the starting point, and the target of a path is the end point. With this set up, the following calculation is a simple solution to this problem:

$$
\begin{aligned}
[f] \star [\bar{g}] &= [e_{x_0}] \\
[f] \star [\bar{g}] \star [g] &= [e_{x_0}] \star [g] \\
[f] &= [g]
\end{aligned}
$$

5. Prove that the subspace $S^1 \times \{x_0\}$ is a retract of $S^1 \times S^1$ but is not a deformation retract, for any point $x_0 \in T$.

Solution: The retraction is given by $r(x, y) = (x, x_0)$. If it were a deformation retraction, then the circle S^1 and the torus $S^1 \times S^1$ would be homotopy equivalent. But they're not since they have non-isomorphic fundamental groups.