Formal vs Informal Definition of a Limit

**Definition 1** (Informal definition). The limit of a function \( f(x) \), as \( x \) approaches \( a \in \mathbb{R} \), is \( L \in \mathbb{R} \), and we write

\[
\lim_{x \to a} f(x) = L
\]

if the values of \( f(x) \) can be made arbitrarily close to \( L \) by choosing the values of \( x \) close enough to \( a \).

**Definition 2** (Formal definition). We write

\[
\lim_{x \to a} f(x) = L
\]

if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon.
\]

Computing limits by definition

To prove that

\[
\lim_{x \to a} f(x) = L
\]

consider a game in which your opponent makes a move by giving you a number \( \varepsilon > 0 \) and you respond by producing a number \( \delta > 0 \) (which depends on \( \varepsilon \)) such that

\[
|x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon
\]

**Example 1.** Prove that

\[
\lim_{x \to 1} 1 = 1
\]

Solution. We have \( f(x) = 1 \) for every \( x \). Suppose \( \varepsilon > 0 \) is given and we have to find \( \delta > 0 \) such that \( |x - 1| < \delta \) implies \( |1 - 1| < \varepsilon \).

In this case any positive number \( \delta \) works since \( |1 - 1| = 0 < \varepsilon \). So, choose \( \delta = 1 \) - this is our response to any given \( \varepsilon \).

Computing limits by definition

**Example 2.** Prove that

\[
\lim_{x \to 1} x = 1
\]

**Example 3.** Prove that

\[
\lim_{x \to 0} x^3 = 0
\]

One-sided limits

Similarly, one can defined one-sided limits as follows
Definition 3. We write 
\[
\lim_{{x \to a^-}} f(x) = L
\]
if the values of \( f(x) \) can be made arbitrarily close to \( L \) by choosing the values of \( x \) close enough to \( a \) and less than \( a \).

We write 
\[
\lim_{{x \to a^+}} f(x) = L
\]
if the values of \( f(x) \) can be made arbitrarily close to \( L \) by choosing the values of \( x \) close enough to \( a \) and greater than \( a \).

One-sided limits

Fact 1.
\[
\lim_{{x \to a^-}} f(x) = L \quad \text{if and only if} \quad \lim_{{x \to a^-}} f(x) = L \quad \text{and} \quad \lim_{{x \to a^+}} f(x) = L.
\]

Example 4.
\[
\lim_{{x \to 0}} \frac{|x|}{x} = \text{undefined}
\]

Basic techniques

Fact 2. The limit of a sum, difference, or product is equal to the sum, difference, or product of the limits provided all the limits involved exist. The limit of a quotient is the quotient of the limits provided that the limit of the denominator is not 0.

Example 5 (Positive example).
\[
\lim_{{x \to 0}} \frac{x^3 + 2x^2 - 1}{1 - 3x} = -1
\]

Example 6 (Negative example).
\[
\lim_{{x \to 2}} \frac{x^2 - 4}{x - 2} = 4
\]

Basic techniques

Fact 3. If \( f(x) = g(x) \) for every \( x \) near \( a \) (but not equal to \( a \)) then
\[
\lim_{{x \to a}} f(x) = \lim_{{x \to a}} g(x)
\]
provided both exist.
So, technically, in the previous example the function \( f(x) = \frac{x^2 - 4}{x^2 - 2} \) is being replaced with the function \( g(x) = x + 2 \) whose limit is easy to compute as \( x \to 2 \).

**Example 7.**

\[
\lim_{x \to 0} \frac{(x + 2)^2 - 4}{x} = 4
\]

Basic techniques

**Example 8.**

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x^2} = \frac{1}{4}
\]

**Example 9.** Compute \( \lim_{x \to 0} f(x) \) if

\[
f(x) = \begin{cases} \sqrt{x} + 2 & \text{if } x \geq 0, \\ 2 - x^2 & \text{if } x < 0 \end{cases}
\]

Basic techniques

**Fact 4.** If \( f(x) \leq g(x) \) for every \( x \) near \( a \) (except possibly at \( a \)) and the limits of \( f(x) \) and \( g(x) \) both exist as \( x \to a \) then

\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
\]

**Fact 5** (Squeeze Theorem). If \( f(x) \leq g(x) \leq h(x) \) for every \( x \) near \( a \) (except possibly at \( a \)) and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \) then

\[
\lim_{x \to a} g(x) = L
\]

**Example 10.** Show that \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \).

**Infinite limits**

**Definition 4.** We write \( \lim_{x \to a} f(x) = \infty \) if the values of \( f(x) \) can be made arbitrarily large by choosing the values of \( x \) close enough to \( a \).

We write \( \lim_{x \to a} f(x) = -\infty \) if the values of \( f(x) \) can be made arbitrarily large negative by choosing the values of \( x \) close enough to \( a \).

One-sided infinite limits are defined similarly.

**Definition 5.** \( x = a \) is called a vertical asymptote for \( f(x) \) if at least one one-sided limit of \( f(x) \) as \( x \) approaches \( a \) is equal to \( \pm \infty \).

**Infinite limits**
Example 11.
\[ \lim_{x \to 1^-} \frac{1}{x - 1} = -\infty \quad \lim_{x \to 1^+} \frac{1}{x - 1} = \infty \]

Example 12.
\[ \lim_{x \to 0^+} \ln x = -\infty \]

Example 13.
\[ \lim_{x \to \frac{\pi}{2}^-} \tan x = \infty \]

Limits at infinity

Definition 6. We write
\[ \lim_{x \to \infty} f(x) = L \]
if the values of \( f(x) \) can be made arbitrarily close to \( L \) by choosing the values of \( x \) large enough.

We write
\[ \lim_{x \to -\infty} f(x) = L \]
if the values of \( f(x) \) can be made arbitrarily close to \( L \) by choosing the values of \( x \) large enough negative.

In either case we say that the horizontal line \( y = L \) is a horizontal asymptote for \( f(x) \).

Limits at infinity

Fact 6. If \( r > 0 \) then
\[ \lim_{x \to \infty} \frac{1}{x^r} = 0, \quad \lim_{x \to -\infty} \frac{1}{x^r} = 0 \]

Example 14.
\[ \lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1 \]

Limits at infinity

Example 15.
\[ \lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = \frac{1}{3} \]

Example 16.
\[ \lim_{x \to -\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = -\frac{1}{3} \]

Limits at infinity
Example 17. 
\[ \lim_{x \to -\infty} e^x = 0 \]

Example 18. 
\[ \lim_{x \to -\infty} \frac{x^2}{e^x} = 0 \]

Example 19. 
\[ \lim_{x \to -\infty} \sin x = \text{undefined} \]

Continuity at a point

**Definition 7.** A function \( f(x) \) is continuous at \( a \in \mathbb{R} \) if
\[ \lim_{x \to a} f(x) = f(a) \]

**Remark 1.** Continuity of \( f(x) \) at \( a \) implies that \( f(a) \) is defined \((a \in \text{Dom}(f))\), \( \lim_{x \to a} f(x) \) exists, and the equality from the definition holds.

**Definition 8.** If \( f(x) \) is not continuous at \( a \) then it is called discontinuous at \( a \). There are three types of discontinuity.

Types of discontinuity

- Removable discontinuity:
  
  if \( a \notin \text{Dom}(f) \) but it is possible to extend \( f \) to a new function \( \tilde{f} \) such that

  \[ \tilde{f}(x) = \begin{cases} 
  f(x), & \text{if } x \in \text{Dom}(f), \\
  \lim_{x \to a} f(x), & \text{if } x = a.
  \end{cases} \]

  **Example 20.** Check continuity of the function
  
  \[ f(x) = \frac{x^2 - 1}{x - 1} \]

  at the point \( x = 1 \).

Types of discontinuity

- Infinite discontinuity:
  one of the limits \( \lim_{x \to a^-} f(x) \), \( \lim_{x \to a^+} f(x) \) is equal to \( \pm \infty \).
Example 21. Check continuity of the function

\[ f(x) = \frac{1}{x} \]

at the point \( x = 0 \).

Types of discontinuity

- Jump discontinuity:
  
  both \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist but are not equal to each other.

Example 22. Check continuity of the function

\[ f(x) = \frac{|x|}{x} \]

at the point \( x = 0 \).

Continuous functions

Definition 9. \( f(x) \) is continuous on an interval if it is continuous at every point of the interval.

Fact 7. The following functions are continuous on their domains: polynomial, rational, root, trigonometric, inverse trigonometric, exponential, logarithmic functions.

Fact 8. Continuous functions (at a point, or on a given interval) are closed under taking a sum, difference, product, and quotient (whenever the denominator is defined).

Continuous functions

Example 23. Where is the function continuous?

(a)

\[ f(x) = \frac{\ln x + e^x}{x^2 - 1} \]

(b)

\[ f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 1, \\
  \frac{1}{x} & \text{if } 1 < x < 3, \\
  \sqrt{x - 3} & \text{if } x \geq 3,
\end{cases} \]
Example 24. Find the values of the parameter c which make the function continuous everywhere

\[ f(x) = \begin{cases} 
  cx^2 + 2x & \text{if } x < 2, \\
  x^3 - cx & \text{if } x \geq 2, 
\end{cases} \]

Continuous functions

Fact 9 (Composition). If \( f(x) \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \), then

\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) \]

Example 25.

\[ \lim_{x \to 2} \tan^{-1} \left( \frac{x^2 - 4}{3x^2 - 6x} \right) = \tan^{-1} \left( \frac{2}{3} \right) \]

Intermediate Value Theorem

Theorem 1. If \( f(x) \) is continuous on the closed interval \([a, b]\) and \( N \in \mathbb{R} \) is between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \), then there exists \( c \in [a, b] \) such that \( f(c) = N \).

Example 26. Show that there is a root of the equation \( x^4 + x - 3 = 0 \) on the interval \((1, 2)\).