(1) The set of solutions to the system of linear equations
\[ \begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_1 - 3x_2 + x_3 &= 0
\end{align*} \]
is a subspace of \( \mathbb{R}^3 \). Find a basis for this subspace.

(2) Prove that \{0\}, \( \mathbb{R}^2 \) and all lines in \( \mathbb{R}^2 \) through the origin are the only subspaces of \( \mathbb{R}^2 \). (Hint: Use a dimension argument.)

(3) Let \( V \) be a vector space, and let \( T \in \mathcal{L}(V) \). A subspace \( W \subseteq V \) is \( T \)-invariant if \( Tx \in W \) \( \forall x \in W \), that is, \( T(W) \subseteq W \). Prove that the subspaces \{0\}, \( V \), \( \text{ran}(T) \) and \( \text{ker}(T) \) are all \( T \)-invariant.

(4) Prove that there does not exist a linear map \( T \) from \( \mathbb{R}^5 \) to \( \mathbb{R}^2 \) such that
\[ \text{ker}(T) = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5 \} \]
Hint: Use the Rank-Nullity Theorem.

(5) Suppose \( T \in \mathcal{L}(V, W) \) is injective and \( \{v_1, \ldots, v_n\} \) is linearly independent in \( V \). Prove that \( \{Tv_1, \ldots, Tv_n\} \) is linearly independent in \( W \).

(6) Extra Credit: A function \( T : V \to W \) between vector spaces \( V \) and \( W \) is called additive if \( T(x + y) = Tx + Ty \) for all \( x, y \in V \). Prove that if \( V \) and \( W \) are defined over \( \mathbb{Q} \), then any additive function from \( V \) to \( W \) is a linear map.

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Friedberg p. 55 #3, #11, #12
   p. 74-75 #3, #5, #9
   p. 84, 86 #2, #13

Prove that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if \( \dim V = 1 \) and \( T \in \mathcal{L}(V) \), then there exists \( \lambda \in F \) such that \( Tv = \lambda v \) for all \( v \in V \).

Consider \( \mathbb{C} \) as a complex vector space. Is complex conjugation a linear operator on \( \mathbb{C} \), i.e. if \( \forall a + bi \in \mathbb{C}, T(a + bi) = a - bi \), does \( T \in \mathcal{L}(\mathbb{C}) \)?