1. Let $V$ be a vector space, and let $T \in \mathcal{L}(V)$. Prove that $T^2 = T_0$ if and only if $\text{ran}(T) \subseteq \text{ker}(T)$.

Proof. We first show that if $T^2 = T_0$, then $\text{ran}(T) \subseteq \text{ker}(T)$. Let $v \in V$ and suppose $T^2 = T_0$. Then, $T^2v = T_0v = 0$. Thus, $Tv \in \text{ker}(T)$. Since $Tv \in \text{ran}(T)$ by definition, $\text{ran}(T) \subseteq \text{ker}(T)$.

We now show the converse. Let $v \in V$ and suppose $\text{ran}(T) \subseteq \text{ker}(T)$. Then, $Tv \in \text{ran}(T) \implies Tv \in \text{ker}(T)$. Then, $T(Tv) = 0 \implies T^2v = 0$. Thus, $T^2v = T_0v = 0 \ \forall v \in V$. □

2. Suppose $V$ and $W$ are both finite-dimensional vector spaces. Prove that there exists a surjective linear map from $V$ to $W$ if and only if $\dim W \leq \dim V$.

Proof. We first show that if there exists a surjective linear map from $V$ to $W$, then $\dim W \leq \dim V$. Let $T \in \mathcal{L}(V,W)$ be surjective. Then, $\text{rank } T = \dim W$. By the Rank-Nullity Theorem,

$$\dim V = \text{nullity } T + \text{rank } T \geq \text{rank } T \geq \dim W.$$  

We now show the converse. Suppose $\dim W \leq \dim V$. Let $\beta = \{\beta_1, \ldots, \beta_n\}$ be a basis for $V$ and $\gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a basis for $W$, with $m \leq n$. Since we have at least as many $\beta$’s as $\gamma$’s, each $\gamma_i$ can be associated with at least one $\beta_j$ by a function $T$ from $V$ to $W$. Then, $T$ must be linear since $T$ is determined by its action on the basis vectors of $V$. $T$ is also surjective since every basis vector of the codomain $W$ has a preimage in $V$. □

3. Let $V$ and $W$ be finite dimensional vector spaces and $T : V \rightarrow W$ be an isomorphism. Let $V_0$ be a subspace of $V$.

a). Prove that $T(V_0)$ is a subspace of $W$.

Proof. Let $V$ and $W$ be finite dimensional vector spaces, and let $T : V \rightarrow W$ be an isomorphism. Let $V_0$ be a subspace of $V$. We wish to show that $T(V_0)$ is a subspace of $W$. To show that a subset is a subspace, we need to show it satisfies the three conditions: i) it contains the zero vector of $W$, $0_W$; ii) it’s closed under addition; and iii) it’s closed under scalar multiplication.

(1) Since $V_0$ is a subspace of $V$, it contains $0_V$. Since $T$ is linear, $T(0_V) = 0_W \in T(V_0)$. 

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(2) Let \( x, y \in T(V_0) \). Then, there exist vectors \( u, v \in V_0 \) such that \( Tu = x \) and \( Tv = y \). Then, \( x + y = Tu + Tv = T(u + v) \in T(V_0) \) since \( u + v \in V_0 \).

(3) Let \( c \in F \) and \( x \in T(V_0) \). Then, there exists a vector \( u \in V_0 \) such that \( Tu = x \). Thus, \( cx = cTu = T(cu) \in T(V_0) \) since \( cu \in V_0 \).

Thus, \( T(V_0) \) is a subspace of \( W \). \( \square \)

b). Prove that \( \dim(V_0) = \dim(T(V_0)) \).

**Proof.** Let \( \beta = \{v_1, \ldots, v_k\} \) be a basis for \( V_0 \). Then, \( \beta \) is linearly independent in \( V \). Since \( T \) is an isomorphism, it’s injective. In HW # 3, problem 5, we proved that the images of linearly independent vectors under an injective linear map must also be linearly independent. Thus, \( \gamma = \{Tv_1, \ldots, Tv_k\} \) is linearly independent in \( W \). Clearly, \( Tv_i \in T(V_0) \) since \( v_i \in V_0 \). So, \( \gamma \subset T(V_0) \). We wish to show that \( \gamma \) is a basis for \( T(V_0) \). Since we’ve just shown that it is linearly independent, we only need to show that \( \gamma \) is also a spanning set for \( T(V_0) \).

Let \( w \in T(V_0) \). Then, there exists \( v \in V_0 \) such that \( Tv = w \). Since \( \beta \) is a basis for \( V_0 \), there exist scalars \( a_1, \ldots, a_k \) such that

\[
v = a_1v_1 + \cdots + a_kv_k.
\]

Then,

\[
w = Tv = T(a_1v_1 + \cdots + a_kv_k) = a_1Tv_1 + \cdots + a_kTv_k
\]

Thus, an arbitrary vector \( w \) in \( T(V_0) \) can be written as a linear combination of the vectors in \( \gamma \). So, \( \gamma \) is a spanning set for \( T(V_0) \). Thus, \( \gamma \) is a basis for \( T(V_0) \), and \( \dim(T(V_0)) = k = \dim(V_0) \). \( \square \)

4. Prove that \( \mathbb{C} \cong \mathbb{R}^2 \) as real vector spaces and use this to show that \( \dim(\mathbb{C}) = 2 \).

**Proof.** We wish to prove first that \( \mathbb{C} \cong \mathbb{R}^2 \) as vector spaces over \( \mathbb{R} \). We will first define a map \( T : \mathbb{C} \to \mathbb{R}^2 \) and show that this map is linear and invertible. Let \( a + ib \in \mathbb{C} \). Then, define \( T(a + ib) = (a, b) \), with \( a, b \in \mathbb{R} \) and \( i = \sqrt{-1} \), i.e. map the real part of the complex number to the first coordinate in the ordered pair, and the imaginary part to the second coordinate. To show that \( T \) is linear, it must satisfy both additivity and homogeneity. Let \( a + ib, c + id \in \mathbb{C} \) and \( k \in \mathbb{R} \). Then,

\[
T(k(a + ib) + (c + id)) = T(ka + ikb + c + id)
\]

\[
= T((ka + c) + i(kb + d))
\]

\[
= (ka + c, kb + d)
\]

\[
= (ka, kb) + (c, d)
\]

\[
= k(a, b) + (c, d)
\]

\[
= kT(a + ib) + T(c + id)
\]
Thus, $T$ is linear. We now show that $T$ is both injective and surjective, which will show that $T$ is an isomorphism.

To show injectivity, we show that $\ker(T)$ is trivial. Consider $T(a + ib) = (0, 0)$. Then, $(a, b) = (0, 0) \implies a = 0, b = 0$. Thus, only the zero vector of $\mathbb{C}$ maps to the zero vector in $\mathbb{R}^2$. Thus, $T$ is injective. To show surjectivity, we need to show that every ordered pair of real numbers can be written as a complex number. This is clearly possible: for any $(a, b) \in \mathbb{R}^2$, let $z = a + ib \in \mathbb{C}$. Thus, $T$ is surjective, and is an isomorphism. Thus, $\mathbb{C} \cong \mathbb{R}^2$. Since isomorphic vector spaces have equal dimension, $\dim(\mathbb{C}) = \dim(\mathbb{R}^2) = 2$. $\square$