1. Find the change of coordinate matrix \( Q \) that changes \( \beta \)-coordinates into \( \gamma \)-coordinates: \( \beta = \{(2, 1), (-4, 1)\} \) and \( \gamma = \{(-4, 3), (2, -1)\} \).

**Proof.** The change of coordinates matrix that changes \( \beta \)-coordinates into \( \gamma \)-coordinates is \( Q = [\text{Id}]_\gamma^\beta \), where Id is the identity map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). So, to find \( Q \), just find the matrix representation of \( \text{Id} \) with respect to \( \beta \) and \( \gamma \). We first compute the images of each basis vector in \( \beta \) and look at their representations in \( \gamma \).

\[
\text{Id}(2, 1) = (2, 1) = a(-4, 3) + b(2, -1) = (-4a + 2b, 3a - b)
\]

\[
\text{Id}(-4, 1) = (-4, 1) = c(-4, 3) + d(2, -1) = (-4c + 2d, 3c - d)
\]

So, we must find the coefficients of the linear combination of vectors in \( \beta \) that represent the image of each basis vector in \( \beta' \). So, we get the following systems:

\[
\begin{align*}
2 &= -4a + 2b \\
1 &= 3a - b
\end{align*}
\]

Solving this system yields \( a = 2 \) and \( b = 5 \). The second system is

\[
\begin{align*}
-4 &= -4c + 2d \\
1 &= 3c - d
\end{align*}
\]

Solving this system yields \( c = -1 \) and \( d = -4 \). Each 2-tuple of coefficients represents a coordinate vector. To get the matrix, we simply place the coordinate vectors next to each other in the matrix:

\[
Q = [\text{Id}]_\gamma^\beta = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}
\]

\( \Box \)

2. Suppose \( \phi \in \mathcal{L}(V, F) \) and \( \phi \) is not the zero map. Prove that \( \dim V/\ker(\phi) = 1 \). Note: \( F \) is the ground field for \( V \).

**Proof.** Since \( \phi \) is not the zero map, \( \text{ran}(\phi) \) contains at least one non-zero element. So, its dimension must be at least 1. We note that \( \dim(F) = 1 \). Since \( \text{ran}(\phi) \) is a subspace of \( F \), \( \dim(\text{ran}(\phi)) \) cannot be more than 1. Thus, \( \dim(\text{ran}(\phi)) = 1 \).

By the First Isomorphism Theorem for Vector Spaces,

\[
V/\ker(\phi) \cong \text{ran}(\phi).
\]

Thus, \( \dim(V/\ker(\phi)) = \dim(\text{ran}(\phi)) = 1 \). \( \Box \)
3. **Extra credit:** Suppose $T \in L(V,W)$ and $U$ is a subspace of $V$. Let $\pi$ be the quotient map from $V$ onto $V/U$. Prove that there exists $S \in L(V/U,W)$ such that $T = S \circ \pi$ if and only if $U \subset \ker(T)$.

*Hint:* For the backwards direction, define the map $S$ explicitly and show that this map must be linear and satisfy the given relationship.

*Proof.* We first prove that if there exists $S \in L(V/U,W)$ such that $T = S \circ \pi$, then $U \subset \ker(T)$.

Suppose there exists $S \in L(V/U,W)$ such that $T = S \circ \pi$. Let $u \in U$. We will show that $u \in \ker(T)$. Since $u \in U$, $u \in V$ and $\pi(u) = [u] = [0_V] \in V/U$. Since $S$ is linear, $S([u]) = S([0_V]) = 0_W$. Then, $0_W = S([u]) = S(\pi(u)) = Tu$. Thus, $u \in \ker(T)$.

Now we prove the reverse direction. Suppose $U \subset \ker(T)$. We will explicitly construct a map $S$ from $V/U$ to $W$ that satisfies the necessary condition. Define $S : V/U \to W$ as $S([v]) = Tv$, where $[v]$ is an arbitrary affine subspace in $V/U$. We first show that $S$ is linear; that is, $S$ satisfies additivity and homogeneity.

Let $[v_1], [v_2] \in V/U$ and $c \in F$. Then,

\[
S(c[v_1] + [v_2]) = S([cv_1] + [v_2]) \\
= S([cv_1 + v_2]) \\
= T(cv_1 + v_2) \\
= cTv_1 + Tv_2 \\
= cS([v_1]) + S([v_2])
\]

Now, we check whether $T = S \circ \pi$ is true for all vectors $v \in V$. Let $v_1$ and $v_2$ be two distinct vectors in $V$ and suppose $[v_1] = [v_2]$. Then, $S(\pi(v_1)) = S([v_1]) = Tv_1$ and $S(\pi(v_2)) = S([v_2]) = Tv_2$. Since $[v_1] = [v_2]$, we need to check whether $Tv_1 = Tv_2$ since we don’t know whether $T$ is injective. Since $[v_1] = [v_2]$, $v_1 - v_2 \in U \implies v_1 - v_2 \in \ker(T)$. Thus, $T(v_1 - v_2) = 0_W \implies Tv_1 = Tv_2$. \(\square\)