Hand In

(1) Let $T \in \mathcal{L}(P_2(\mathbb{R}))$ such that $T(f(x)) = xf'(x) + f(2)x + f(3)$. Find the eigenvalues of $T$ and an eigenbasis $\beta_E$ for $V$, if one exists.

(2) Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda$ is a nonzero scalar. Prove that if $\lambda$ is an eigenvalue of $T$, then $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

(3) Let $T \in \mathcal{L}(V)$. Prove that $T$ is invertible if and only if $0$ is not an eigenvalue of $T$.

(4) Suppose $V$ is finite-dimensional and $v_1, \ldots, v_m \in V$. Prove that $\{v_1, \ldots, v_m\}$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of $T$ corresponding to distinct eigenvalues.

DNHI

Friedberg p. 257 #4a, 4b, 4h
p. 280 #3d, #8

The Fibonacci sequence is the classic sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$ where each term in the sequence is the sum of the two preceding terms, i.e. $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It’s an example of a recursive sequence. For example, if we wanted to find the 75th term in the sequence, we’d need the 74th and the 73rd term to do so, and the process repeats.

However, using eigenvalues and eigenvectors, we can derive an explicit formula for the $n$th term in the Fibonacci sequence. Define $T \in \mathcal{L}(\mathbb{R}^2)$ as $T(x,y) = (y, x+y)$.

(1) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each positive integer $n$.

(2) Find the eigenvalues of $T$.

(3) Find an eigenbasis for $\mathbb{R}^2$.

(4) Use this eigenbasis to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right],$$

for each positive integer $n$.  

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Note: $T^n$ is just $T$ composed with itself $n$ times.