1. Let $T \in \mathcal{L}(P_2(\mathbb{R}))$ such that $T(f(x)) = xf'(x) + f(2)x + f(3)$. Find the eigenvalues of $T$ and an eigenbasis $\beta$ for $V$.

Proof. Let $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$. We wish to compute the eigenvalues of $T$ and give an eigenbasis to diagonalize $T$. We start with the standard basis for $P_2(\mathbb{R})$ to get a matrix representation of $T$, $\beta_{\text{std}} = \{1, x, x^2\}$. Then, we get the following image vectors:

\[
\begin{align*}
T(1) &= 0 + 1x + 1 = x + 1 \\
T(x) &= x + 2x + 3 = 3x + 3 \\
T(x^2) &= 2x^2 + 4x + 9
\end{align*}
\]

This gives the following matrix representation for $T$:

\[
A = [T]_{\beta_{\text{std}}} = \begin{pmatrix}
1 & 3 & 9 \\
1 & 3 & 4 \\
0 & 0 & 2
\end{pmatrix}
\]

Now, we can compute the characteristic polynomial of $A$, which is equal to the characteristic polynomial of $T$. Since $\text{char } A = \det(A - tI_3)$, we have

\[
A - tI_3 = \begin{pmatrix}
1 - t & 3 & 9 \\
1 & 3 - t & 4 \\
0 & 0 & 2 - t
\end{pmatrix}
\]

Then, $\text{char } A = -t(t-2)(t-4)$. Setting $\text{char } A = 0$, we get the roots of this polynomial, $t = 0, 2, 4$, which are the eigenvalues of $A$ (and $T$). Note that since the dimension of $V$ is 3 and we have 3 distinct eigenvalues, $T$ (and $A$) is diagonalizable, so an eigenbasis exists. We now compute the eigenvectors associated to each eigenvalue.

Recall that a nonzero vector $v$ is an eigenvector associated to $\lambda$ if and only if $v \in \ker(T - \lambda I_d)$. This is the same as plugging in each eigenvalue into (1) for $t$, and solving the system of equations to get the eigenvectors.

Note: The eigenvectors we will get here are eigenvectors for $A$, not $T$. We have to use the isomorphism between $V$ and $\mathbb{R}^3$ to "lift" the eigenvectors of $A"$ "upstairs" to the eigenvectors of $T$ (recall the commutative diagram with $T$ and $L_A$).
For $t = 0$, let $w = (w_1, w_2, w_3)$ be the associated eigenvector. Then,
\[
(A - 0I_3)x = 0
\]
\[
\begin{pmatrix}
1 & 3 & 9 \\
1 & 3 & 4 \\
0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} = \begin{pmatrix}0 \\
0 \\
0\end{pmatrix}
\]
So, we have the following system of equations:
\[
w_1 + 3w_2 + 9w_3 = 0 \\
w_1 + 3w_2 + 4w_3 = 0 \\
2w_3 = 0
\]
Then, we get $w_3 = 0$ and $w_1 = -3w_2$, which means we can take $w_2$ to be the free variable. So, an eigenvector associated to $\lambda = 0$ is $(-3, 1, 0)$.

For $t = 2$, let $y = (y_1, y_2, y_3)$ be the associated eigenvector. Then,
\[
(A - 2I_3)x = 0
\]
\[
\begin{pmatrix}
-1 & 3 & 9 \\
1 & 1 & 4 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}0 \\
0 \\
0\end{pmatrix}
\]
So, we have the following system of equations:
\[
-y_1 + 3y_2 + 9y_3 = 0 \\
y_1 + y_2 + 4y_3 = 0
\]
Then, we get $y_2 = -\frac{13}{4}y_3$ and $y_1 = -\frac{3}{4}y_3$, so $y_3$ is the free variable. So, an eigenvector associated to $\lambda = 2$ is $(-3, -13, 4)$.

For $t = 4$, let $z = (z_1, z_2, z_3)$ be the associated eigenvector. Then,
\[
(A - 4I_3)x = 0
\]
\[
\begin{pmatrix}
-3 & 3 & 9 \\
1 & -1 & 4 \\
0 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix} = \begin{pmatrix}0 \\
0 \\
0\end{pmatrix}
\]
So, we have the following system of equations:
\[
-3z_1 + 3z_2 + 9z_3 = 0 \\
z_1 - z_2 + 4z_3 = 0 \\
-2z_3 = 0
\]
Then, we get $z_3 = 0$ and $z_1 = z_2$, taking $z_2$ to be the free variable. So, an eigenvector associated to $\lambda = 4$ is $(1, 1, 0)$. 

Now, we need to get the eigenvectors of $T$ by using the standard isomorphism between $V$ and $\mathbb{R}^3$. So, $w$ in $\mathbb{R}^3$ becomes $x - 3$, $y$ becomes $4x^2 - 13x - 3$ and $z$ becomes $x + 1$. Thus, we have an eigenbasis for $T$: $\beta = \{x - 3, 4x^2 - 13x - 3, x + 1\}$. □

2. Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda$ is a nonzero scalar. Prove that if $\lambda$ is an eigenvalue of $T$, then $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

Proof. Let $T \in \mathcal{L}(V)$ be invertible and $\lambda$ be a nonzero scalar. Suppose $\lambda$ is an eigenvalue of $T$. Then, there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. Since $T$ is invertible, apply $T^{-1}$ to both sides:

$$T^{-1}(Tv) = T^{-1}(\lambda v)$$

$$v = \lambda(T^{-1}(v))$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$.

□

3. Let $T \in \mathcal{L}(V)$. Prove that $T$ is invertible if and only if 0 is not an eigenvalue of $T$.

Proof. We will prove the contrapositive, and prove both directions simultaneously. Let $\beta$ be a basis of $V$ and let $A = [T]_\beta$. Then, $\lambda = 0$ is an eigenvalue of $T$ $\iff$ $\det(A - \lambda I_n) = \det(A - 0I_n) = 0$ $\iff$ $\det(A) = 0$ $\iff$ $A$ is not invertible $\iff$ $T$ is not invertible. □

4. Suppose $V$ is finite-dimensional and $v_1, \ldots, v_m \in V$. Prove that $\{v_1, \ldots, v_m\}$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of $T$ corresponding to distinct eigenvalues.

Proof. Note that the reverse direction was already proven in class by induction: Eigenvectors corresponding to distinct eigenvalues of a linear operator $T$ must be linearly independent. So, we only have to prove the forward direction.

Suppose $\{v_1, \ldots, v_m\}$ is linearly independent. Then, we can extend this to a basis of $V$: $\{v_1, \ldots, v_m, w_1, \ldots, w_l\}$. Pick distinct scalars $\lambda_1, \ldots, \lambda_m$ and define a function $T: V \rightarrow V$ such that $Tv_i = \lambda_i v_i$ and $Tw_j = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq l$. Since a linear map is defined by its action on a basis, $T$ must be linear. Since $Tv_i = \lambda_i v_i$ and $\lambda_i$ are all distinct, $v_1, \ldots, v_m$ are eigenvectors of $T$ corresponding to $\lambda_1, \ldots, \lambda_m$, respectively. □