Topology II Notes
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1. Definition and Basic Properties of Homotopy Groups

Let $e_0 = \{1,0,\ldots,0\} \in S^n$. $[X,Y]$ denotes homotopy equivalence classes of maps (similarly for pairs, triples, etc.).

**Definition 1.1.** For a pointed space $(X,x_0)$, define the $n$th homotopy group to be

$$\pi_n(X,x_0) = [(S^n,e_0),(X,x_0)].$$

**Definition 1.2.** For a pointed pair of spaces $(X,A,x_0)$ define the $n$th relative homotopy group to be

$$\pi_n(X,A,x_0) = [(D^n,S^{n-1},e_0),(X,A,x_0)].$$

**Proposition 1.3.** For a pointed pair $(X,A,x_0)$ there is a long exact sequence

$$\cdots \to \pi_{n+1}(X,A,x_0) \to \pi_n(A,x_0) \to \pi_n(X,x_0) \to \pi_n(X,A,x_0) \to \cdots$$

**Proposition 1.4.** Let $(X,A,x_0)$ be a pointed pair. Given a strictly commutative diagram

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow f|_{S^{r-1}} & & \downarrow f \\
S^{r-1} & \longrightarrow & D^r
\end{array}$$

the map $f : (D^r,S^{r-1},e_0) \to (X,A,x_0)$ represents the identity element in $\pi_r(X,A,x_0)$ if and only if $f$ is homotopic rel $S^{r-1}$ to a map $f' : D^r \to A$.

2. Cofibrations

**Proposition 1.5.** Let $(Y,B)$ be a pointed pair such that the inclusion $i : B \hookrightarrow Y$ is an $n$-equivalence. Suppose there is a diagram

$$\begin{array}{ccc}
B & \longrightarrow & Y \\
\downarrow g & & \downarrow f \\
S^{r-1} & \longrightarrow & D^r
\end{array}$$
with \( r \leq n \) which commutes up to homotopy. Let \( H : S^{r-1} \times I \to Y \) be a homotopy between \( f|_{S^{r-1}} \) and \( i \circ g \). There exists a map \( f' : D^r \to B \) such that \( f'|_{S^{r-1}} = g \) and \( H \) extends to a homotopy between \( f \) and \( i \circ f' \).

**Proof.** We glue the map \( f \) together with the homotopy \( H \) to get a strictly commuting square. Specifically, define \( \bar{f} : D^r \to Y \) as follows:

\[
\bar{f}(x) = \begin{cases} 
  f(2x) & \text{if } 0 \leq |x| \leq 1/2 \\
  H \left( \frac{x}{|x|}, 2|x| - 1 \right) & \text{if } 1/2 \leq |x| \leq 1
\end{cases}
\]

Then the following square is strictly commuting:

\[
\begin{array}{ccc}
B & \xrightarrow{i} & Y \\
\downarrow{g} & & \downarrow{\bar{f}} \\
S^{r-1} & \xleftarrow{i} & D^r
\end{array}
\]

Apply Proposition 1.4 to this square to get a map \( f' : D^r \to B \) with \( \bar{f} \) homotopic to \( i \circ f' \). \( f \) is homotopic to \( \bar{f} \) by \( \bar{f}(x(t+1/2)) \), so we get \( f \) homotopic to \( i \circ f' \) by a homotopy which extends the homotopy \( H \).

**Theorem 1.6.** Let \( (X, A) \) be a CW pair, and let \( (Y, B) \) be a pair such that the inclusion \( i : B \to Y \) is a weak homotopy equivalence. Suppose we have a strictly commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i} & Y \\
\downarrow{f|_A} & & \downarrow{f} \\
A & \xleftarrow{i} & X
\end{array}
\]

Then \( f \) is homotopic rel \( A \) to \( i \circ f' \) for some map \( f' : X \to B \).

**Proof.** We will use Zorn’s Lemma. Consider the collection \( \mathcal{L} \) of CW complexes between \( A \) and \( X \) for which the conclusion of the Proposition holds. In other words, let \( \mathcal{L} \) be the collection of triples \( (U, g', G) \) where \( U \) is a CW complex with \( A \subset U \subset X \), \( g' : U \to B \) with \( g'|_A = f|_A \), and \( G : U \times I \to Y \) is a homotopy rel \( A \) between \( f|_U \) and \( i \circ g' \). \( \mathcal{L} \) is not empty since \((A, f|_A, G) \in \mathcal{L}\), where \( G \) is the constant homotopy. \( \mathcal{L} \) is partially ordered by inclusion and all chains have upper bounds (just take the union), hence a maximal element \((U, g', G) \in \mathcal{L} \) exists. We need to see why \( U = X \).

If not then let \( e \) be an \( r \)-cell in \( X \) not in \( U \). Assume \( r \) is minimal among such cells, so that \( U \cup e \) is obtained by attaching \( D^r \) to \( U \) with
characteristic map \( \Phi : (D^r, S^{r-1}) \to (U \cup e, U) \). Apply Proposition 1.5 to
\[
\begin{array}{c}
B \\
\downarrow ^{i}
\end{array}
\begin{array}{c}
Y
\end{array}
\begin{array}{c}
f \circ \Phi
\end{array}
\begin{array}{c}
\downarrow \\
\uparrow ^{g' \circ \Phi |_{S^{r-1}}}
\end{array}
\begin{array}{c}
S^{r-1}
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
D^r
\end{array}
\]
to obtain a map \( D^r \to B \) and a homotopy \( H : D^r \times I \to Y \) between \( f \circ \Phi \) and \( g' \circ \Phi |_{S^{r-1}} \). This homotopy, together with the given homotopy on \( U \), defines a homotopy on \( U \cup e \) making \( U \cup e \) an element of \( \mathcal{L} \) which contradicts the maximality of \( U \). Therefore we must have \( U = X \).

3. Fibrations

4. Hurewicz Theorems and Whitehead Theorems

**Theorem 1.7.** Let \( f : X \to Y \) be weak homotopy equivalence. Then for any CW complex \( K \), \( [K,X] \xrightarrow{f_*} [K,Y] \) is a bijection.

**Proof.** By appealing to the mapping cylinder of \( f \), we can assume, without loss of generality, that the map \( f \) is injective.

- **surjective:** Let \([a] \in [K,Y]\). Apply Theorem 1.6 to get a map \( a' : K \to X \) such that \( f \circ a' \) is homotopic rel \( \{k_0\} \) to \( a \), i.e. \( f_*([a']) = [a] \).
- **injective:** Let \([a_0],[a_1] \in [K,X]\) such that \( f_*([a_0]) = f_*([a_1]) \).
  Then there is a homotopy \( H : K \times I \to Y \) such that \( H_i = a_i \), \( i = 0,1 \). Define \( h : K \times \{0\} \cup K \times \{1\} \cup \{k_0\} \times I \to X \) by
  \[
  h(k,t) = \begin{cases} 
  a_0(k) & \text{if } t = 0 \\
  a_1(k) & \text{if } t = 1 \\
  x_0 & \text{if } k = k_0 
  \end{cases}
  \]
  Now apply Theorem 1.6 to
  \[
  \begin{array}{c}
  X \\
  h
  \end{array}
  \begin{array}{c}
  \downarrow ^{f}
  \end{array}
  \begin{array}{c}
  Y
  \end{array}
  \begin{array}{c}
  \downarrow ^{H}
  \end{array}
  \begin{array}{c}
  K \times \{0\} \cup K \times \{1\} \cup \{k_0\} \times I \\
  \end{array}
  \begin{array}{c}
  \rightarrow
  \end{array}
  \begin{array}{c}
  K \times I
  \end{array}
  \]

**Corollary 1.8 (Whitehead Theorem).** Let \( f : X \to Y \) be weak homotopy equivalence between CW complexes. Then \( f \) is a homotopy equivalence.
Proof. Apply Theorem 1.7 to the case $K = Y$ to produce a map $g : Y \to X$ such that $f_*(\langle g \rangle) = \langle 1_Y \rangle$. So $f \circ g \sim 1_Y$. Now let $K = X$ and notice $f_*([g \circ f]) = [f \circ g \circ f] = [f] = f_*(1_X)$. Since $f_*$ is 1-1, we have $g \circ f \sim 1_X$. So $g$ and $f$ are inverse homotopy equivalences. \qed

5. Eilenberg-Mac Lane Spaces and Representability of Classical Cohomology
CHAPTER 2

Spectral Sequences

1. General Construction

2. Example – Serre Spectral Sequence of a Fibration
   2.1. Construction.
   2.2. Convergence.
   2.3. $E_2$-term.

3. Application – Cohomology of E-M Spaces and Cohomology Operations

4. The Steenrod Algebra
2. SPECTRAL SEQUENCES
CHAPTER 3

Stable Homotopy Theory and Spectra

1. Stable Homotopy Groups
2. The Category of Spectra
3. Generalized Homology and Cohomology Theories
3. STABLE HOMOTOPY THEORY AND SPECTRA
CHAPTER 4

The (Stable) Adams Spectral Sequence

1. Construction and Properties

1.1. A Digression – Homological Algebra.

1.2. The $E_2$-term.

1.3. Completions and Localizations.

1.4. Convergence.

2. Applications

2.1. Hopf Invariant One and Vector Fields on Spheres.

2.2. Cobordism Rings.

2.3. Other Applications.