RESEARCH STATEMENT

VINCENT R. MARTINEZ

My main research interests are rooted in the analysis of partial differential equations (PDEs), particularly in hydrodynamic or geophysical equations, such as the incompressible Navier-Stokes, the surface quasi-geostrophic (SQG) equations, and various related equations, such as chemotaxis equations that incorporate the effects from interaction of an organism with an ambient incompressible fluid, and dispersive equations that incorporate weak damping mechanisms. A large source of inspiration derives from the mathematics of turbulence through the characterization of small length scales, manifestations of finite-dimensionality in the long-time regime, as well as its applications, for instance, to dissipative dynamical systems, data assimilation (DA), or parameter estimation. In order treat these various considerations, my work often employs tools and techniques from harmonic analysis, elliptic, parabolic, and hyperbolic equation theory, infinite-dimensional analysis, approximation theory, and control theory. Due to the interdisciplinary nature of my work, it is also accompanied, at times even driven, by computational efforts. Generally speaking, however, my research moves along three interrelated directions: (I) Well-posedness and Regularity, (II) Long-time Behavior: Deterministic and Statistical, (III) Applications to DA and Parameter Estimation. In what follows, I describe selected works in each direction, identify how perspectives from turbulence enter some of them, and mention current investigations and future ones that emanate from them.

1. Well-posedness and Regularity

Well-posedness of the Cauchy initial value problem (IVP) is a fundamental issue in the study of evolutionary equations as it asserts the existence and uniqueness of solutions, as well as continuity with respect to initial data. In the context of hydrodynamics, this most basic form of validation for a physical model has yet to be fully settled for the equations of motion for a three-dimensional (3D), incompressible viscous fluid, that is, the 3D Navier-Stokes equations (NSE). In $\mathbb{R}^d$, $d = 2, 3$, one has

$$\partial_t u - \nu\Delta u + (u\cdot \nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0, \quad u(0, x) = u_0(x), \tag{1.1}$$

where $u = u(t, x)$ denotes the velocity vector field of the fluid, $\nabla\cdot u = 0$ expresses its incompressibility, p is the scalar pressure field, u_0 is the initial velocity field, ν is the kinematic viscosity, and (1.1) is appropriately supplemented wtih decay at infinity. The case $\nu = 0$ corresponds to the Euler equations and is referred to as the *inviscid case*. Although global existence and uniqueness of weak solutions is known in 2D [84], while in 3D, global-in-time existence of weak solutions satisfying an energy inequality [58,84] and local-in-time existence *and* uniqueness of strong solutions to (1.1) is known [38,48,62,75,79], the problem of whether such weak solutions are unique when $\nu > 0$ or if singularities can develop for strong solutions in finite time have been outstanding open problems since the equations were conceived in the 19th century. The latter is known as the *global regularity problem* for the 3D NSE and is listed by the Clay Mathematics Institute alongside the Riemann Hypothesis, the Hodge Conjecture, and several others as one of the great unsolved problems in mathematics. The study of the well-posedness of (1.1) continues to be a rich vein of research and source of mathematical problems, as evidenced by recent breakthroughs in its understanding: non-uniqueness of weak solutions to 3D Euler [14, 33, 61], of 3D NSE [15, 25], loss of continuity in initial data for 3D Euler [12, 40, 41] and 3D NSE [13], and finite-time blow-up of Hölder class solutions to 3D Euler [39].

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Contributions. One arm of my research is dedicated to shedding light on well-posedness and regularity properties of hydrodynamic equations and related systems. These models share some structural similarities to (1.1), but typically enjoy a reduced dimensionality or other regularizing mechanisms. One important family of such models are given by the generalized SQG (gSQG) equations (see [19–21]):

$$\partial_t\theta + \gamma\Lambda^\kappa\theta + (u\cdot\nabla)\theta = 0, \quad u = -\nabla^\perp\Lambda^{\beta-2}\theta, \quad \theta(0,x) = \theta_0(x), \tag{1.2}$$

where $\kappa \in (0,1)$, $\beta \in [0,2)$, $\gamma \geq 0$, and $\theta = \theta(t,x)$ represents a scalar quantity. We refer to the case $\gamma = 0$ as the *inviscid* gSQG equation. In the inviscid case, (1.2) interpolates between the 2D Euler equation ($\beta = 0$) and the SQG equation ($\beta = 1$), then extrapolates beyond with increasingly singular constitutive law. A great deal of efforts have gone towards the understanding of the well-posedness of (1.2) since its introduction to the mathematical community by Constantin, Majda, and Tabak [27], especially due to its structural analogy to (1.1) in 3D when case $\beta = 1$ and either $\kappa = 1$ (to NSE) or $\gamma = 0$ (to Euler) (see, for instance, [16, 28–32, 63, 76–78, 89]).

In a series of works with collaborators [8, 64–66], we probe the relation between the dissipation and constitutive law in terms of well-posedness. The choice of functional setting is one of borderline regularity in the sense that it is the lowest level of regularity in which one can expect well-posedness from the perspective of scaling. Due to the non-negative definite nature of the dissipation, smoothing effects also arise. Thus, we also study the extent to which these regularization effects hold in these setting. For this, it is convenient to consider the following generalization to (1.2):

$$\partial_t\theta + \gamma m(D)\theta + (u\cdot\nabla)\theta = 0, \quad u = -\nabla^\perp a(D)\theta, \tag{1.3}$$

where $\mathcal{F}(\mathscr{L}\theta)(\xi) = m(\xi)(\mathcal{F}\theta)(\xi)$, and $m(D), a(D)$ are non-negative, radial multiplier operators.

1. ($m(D) = \Lambda^\kappa, a(D) = \Lambda^{-1}$, $\kappa \in (0,1)$) Then (1.3) possesses a scaling symmetry, $\theta_\lambda = \lambda^{\kappa-1}\theta(\lambda^\kappa t, \lambda x)$. We study the supercritical regime, $\kappa < 1$ in [8] and show that (1.3) has local existence and uniqueness for large initial data in the scaling-critical Besov space, $\dot{B}^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)$, for $2 \leq p < \infty$ and $1 \leq q \leq \infty$, and global existence existence and uniqueness for small initial data. In spite of supercriticality, we show that the maximal spatial regularity arising from the parabolic operator $L = \partial_t + \gamma\Lambda^\kappa$ is conferred instantaneously. In this case, the regularity is captured by *subanalytic Gevrey classes*. In doing so, we extend the technique of Gevrey norms developed by Foias and Temam [46] from L^2–based Sobolev space to L^p–based Besov spaces. Due to supercriticality, commutator estimates are necessary to develop, particularly when carrying out estimates in the Besov–based Gevrey class. We view the commutator as a bilinear multiplier operator and verify a Marcinkiewicz-type condition. Owing to various localizations arising from working in the Besov space setting, we show that this condition is sufficient to obtain $L^p \times L^q \to L^r$–type bounds, although it is well-known that such bilinear operators do not satisfy *any* such bounds in general [52]. Our result also complements ones in the critical space setting previous to this for the 3D NSE [3, 49] and 2D subcritical SQG [36].
2. ($m(D) = \Lambda^\kappa, a(D) = \Lambda^{\beta-2}$, $\beta \in (0,1)$) Again, (1.3) has a scaling symmetry, $\theta_\lambda = \lambda^{\kappa-\beta}\theta(\lambda^\kappa t, \lambda x)$ and we establish results analogous to those in [8], but in the setting of scaling-critical Sobolev spaces, $\dot{H}^{\beta+1-\kappa}(\mathbb{R}^2)$. This provides a direct extension of the results in [4] to the most singular range of the gSQG family. Due to the more singular nature of the constitutive law, the commutator structure of (1.3) is exploited in a crucial and more nuanced way. The commutators identified in [19, 59] allow one to carry out an apriori analysis. However, the scaling-critical setting requires one to identify a suitable approximation procedure to *construct* the solution. Since stability–type estimates in the critical topology for (1.3) are not known, one *cannot* simply carry out a density-type argument with smooth initial data, which would even be a preliminary step for the artificial viscosity system. Our approach overcomes this by proposing a conservation law approximation to the inviscid portion of the system, where the divergence of the flux collapses to the advective nonlinearity in the limit of the approximation. This approximation suitably preserves the underlying commutator structure and ultimately allows for construction solutions to (1.3) at critical regularity. We make an important

observation in identifying an additional structural criticality when $\kappa = \beta - 1$; above this line, (1.3) exhibits a "strongly" quasilinear structure in that the coefficients of the equation have an order that strictly exceeds the order of the linear part. Below this line in the (κ, β)–plane, the approximation procedure is classical, while above this line, the proposed approximation is deployed. We would like to extend this to the L^p setting and eventually carry out the critical space program for 3D NSE to the 2D gSQG system.

3. ($m(D) = 0, a(D) = \log^{\mu}(e - \Delta)\Lambda^{\beta-2}$, $\mu \in (1/2, \infty)$, $\beta \in (1, 2)$) Then (1.3) is inviscid with constitutive law *mildly* regularized by a power of a log-laplacian. In [64], we show that (1.3) is locally well-posed in the borderline Sobolev space, $H^{\beta+1}(\mathbb{R}^2)$, that is, one has existence, uniqueness, as well as continuity with respect to initial data. This extends the work [22], which establishes local well-posednes for $\beta \in [0, 1]$. The most important distinction between $\beta \in [0, 1]$ and $\beta \in (1, 2)$, once again, does *not* arise in the apriori estimates, but rather in stability. Indeed, in [22] continuity with respect to initial data follows by classical means (see [85]). Such an approach fails in range $\beta \in (1, 2)$ and it becomes crucial to exploit the commutator structure of (1.3). We do this by modifying a splitting technique of Kato [74] for symmetric hyperbolic systems that preserves the more nuanced commutator structure of the equation. We would like to explore other forms of mild regularization of systems and attempt to characterize the phenomenon of borderline well-posedness.

2. Long-time Behavior: Deterministic and Statistical

According to the Kolmogorov 1941 theory of 3D turbulence [80, 81], energy cascades from large scales to small scales through a nonlinear mechanism. This cascade should then extend down to the so-called *dissipation length scale*, which indicates the scale at which energy produced from nonlinear interactions is in an exact balance with viscous dissipation. At this length scale, the energy spectrum experiences an exponential drop-off. Specifically, Kolmogorov posited that the start of the dissipation length scale, ℓ_{Kol}, is the length scale uniquely determined by the viscosity and the average rate of energy dissipation in the flow. This length scale represents the smallest relevant scale in the system. From this, Landau and Lifshitz [83] defined the number of degrees of freedom, N_{LL}, to be the total number of eddies of this size that saturate the size of the domain, ℓ_{dom}^3 (if the domain is a box), so that $N_{LL} = (\ell_{dom}/\ell_{Kol})^3$.

There are several other ways for making these notions precise through properties of solutions to (1.1). One such way in light of the observations described above is through *analytic regularity* of the velocity field. Indeed, by a classical harmonic analysis result of Paley and Wiener, exponential decay of the Fourier spectrum of a function characterizes its radius of real analyticity, ℓ_a. Thus, estimates on the analyticity radius of velocity automatically yields an estimate on the corresponding *dissipation wave-number*, $\kappa_a = \ell_a^{-1}$. This subsequently indicates where the energy spectrum begins to experience *exponential decay*, as expected at the dissipation length scale. Hence, the number of degrees of freedom determined by the analyticity radius is then given by $N_a = (\ell_{dom}/\ell_a)^3$. This point of view for estimating the number of degrees of freedom was developed in several works [6, 9, 10, 35, 46, 53, 54, 82].

Another way to define the number of degrees of freedom is through the notion of *enslavement of scales*, which asserts that the dynamics of the flow at small scales or high wave-numbers are enslaved by the low wave-numbers or large scales, respectively, of the flow. It was shown in [44] that solutions to the 2D NSE satisfy an *asymptotic* formulation of this property. Specifically, there exists a minimal wave-number size, N_{dm}, such that any two solutions that are known to converge asymptotically on the finite-dimensional subspace spanned by the eigenmodes of $-\Delta$ up to size N_{dm}, necessarily converge asymptotically to each other. Thus, the Fourier modes corresponding to $|k| > N_{dm}$ are asymptotically enslaved to those corresponding to $|k| \leq N_{dm}$. Since knowledge of the behavior of the solution across finitely many length scales suffice to determine the behavior on all scales, the Fourier modes corresponding to $|k| \leq N_{dm}$ are referred to as *determining modes* and N_{dm} is the *number of determining* *modes*. Significant efforts have been dedicated to estimating the size of N_{dm} and related quantities for the NSE and related models [23, 24, 26, 45, 57, 60, 70–73].

Contributions. A second arm of my research either studies these properties directly or exploits them in order to capture the long-time behavior of solutions to hydrodynamic and related systems that are either deterministically or stochastically perturbed. This is done in a series of works dating back to my Ph.D. thesis [6] and continuing in [50, 55, 56, 68]. I also study forms of stability in other systems arising in chemotaxis [1, 87, 88, 91]. In the following, I describe three works that address the issue of estimating the smallest length scales [6], imbedding the dynamics of (1.2) into that of an ODE [68], as well verifying the ergodic hypothesis for a weakly-damped, stochastically forced system [50].

4. In [6], the Gevrey-norm approach of Foias and Temam [46] is adapted to a Wiener algebra framework, i.e., L^1–based, to obtain refined estimates on the real analyticity radius for the (1.1) over a periodic domain in *any* spatial dimension. As a result, we obtain

$$\ell_a \gtrsim \ell_{Kol}^4 \text{ (in 3D)} \quad \text{and} \quad \ell_a \gtrsim \ell_{Kr}^2 \text{ (in 2D)}, \tag{2.1}$$

for turbulent flows that satisfy (1.1) in a periodic domain, where ℓ_{Kr} refers to the Kraichnan dissipation length scale, which is 2D analog of the Kolmogorov dissipation length scale represented by ℓ_{Kol}. The best-to-date estimates for ℓ_{Kr} in the periodic setting were obtained in [82] resorting to complex-analytic techniques. While the work of [35] established the best-to-date estimates for ℓ_{Kol} in the same setting, but with an L^2–based Gevrey-norm approach in the same setting. These estimates are captured by (2.1). Hence, our result unifies the results of [35] in 3D and [82] in 2D *under a single framework*. Moreover, our method exposes *a new path* to lowering the above exponents closer to 1, namely, by improving higher-order estimates of the flow *in the long-time average*, which would appeal to a statistical framework of turbulence.

5. In the work [68], we establish the existence of a *determining form* (DF) for the subcritical SQG equation, i.e., (1.2), $\kappa \in (1, 2)$ and $\beta = 1$, that is induced by a feedback control system (see (3.2)). A DF is an *ODE*, albeit in an infinite-dimensional Banach space of trajectories, that subsumes the dynamics of the original equation in a certain way. Motivated by the finite dimensionality of the dynamics of solutions to the 2D NSE manifested in the form of the asymptotic enslavement of scales, the notion of DFs was introduced in [43, 47]. A stronger formulation of the notion of enslavement of scales is embodied in the existence of an *inertial form*, (IF) which is a *finite-dimensional system of ODEs* governing the large scale evolution of the evolution that is *fully decoupled* from the small scale evolution. Although solutions to the 2D NSE are known to possess the property of *asymptotic enslavement* [44], the existence of an inertial form for (1.1) in two-dimensions is an open problem.

   In [68], we specifically show that there exist Banach spaces, X, Y, and a map $W : X \to Y$, which is the solution operator of (3.2) corresponding to a "reference solution," $u \in X$, such that $I_hW : B^\rho_X(0) \to Y$ is *Lipschitz*, for some ball of radius $\rho > 0$, centered at 0 in X, where I_h is given by (smooth) projection onto Fourier modes $|k| \leq 2^{1/h}$. It then follows from [47], for instance, that the equation given by

$$\frac{dv(\cdot)}{d\tau}(\tau) = -\|v(\cdot)(\tau) - I_hW(v(\cdot)(\tau))\|_X^2(v(\cdot)(\tau) - I_h\theta^*), \quad v(0) = v_0 \in \mathcal{B}^\rho_X(0). \tag{2.2}$$

   is defined by a right-hand that is Lipschitz and hence, defines an *ODE* in X, where θ^* is a given steady state of (1.2). Our proof of (2.2) defining an ODE hinges on obtaining uniform estimates in L^∞ for solutions to (3.2) *independent* of the number of modes $m \sim 2^{1/h}$. Indeed, the L^p–maximum principle one is able to derive for (3.2) *depends on the number of modes*, which precludes one from using these bounds to establish the Lipschitz property. Nevertheless, by appealing to harmonic analysis tools, we show that one can adapt De Giorgi-type techniques [16] to deduce uniform bounds in L^∞ that are independent of m. The next big step in this program is to directly compare DFs with IFs in cases where IFs exist to study the relationship between these two notions.

6. In the recent work [50], we establish existence and uniqueness of invariant measures for the damped-driven, periodic Korteweg-de Vries (KdV) equation, where a large, but finite number of Fourier modes, $N = N(\gamma) \gg 1$, are stochastically forced:

$$du + (\gamma u + u_{xxx} + uu_x)dt = f dt + \sigma dW, \tag{2.3}$$

where u represents an amplitude, $\gamma > 0$ captures damping effects, f is an external, time-independent deterministic forcing, and $\sigma W = \sum_{j=1}^{N} \sigma_j W_j$ is a Wiener process such that $\sum_{j=1}^{N} \|\sigma_j\|_{H^2}^2 < \infty$ and each $W_j = W_j(t)$ is a 1D standard Brownian motion. A typical assumption made in the study of turbulence is to assume *ergodicity* of its dynamics. This assumption asserts that long-time averages can be equated with ensemble averages, that is, averages over all possible states of the system, and is called the *ergodic hypothesis*. Since ergodic invariant measures form the extremal points of the set of invariant measures, uniqueness of the invariant measure guarantees its ergodicity. In [50], we thus verify the ergodic hypothesis for phenomena described by (2.3).

The propagation of the noise through the system, which is crucial to establishing uniqueness, is well-understood for the 2D NSE [37] and essentially arises due to the balance between heat dissipation and nonlinear effects that ultimately yields the asymptotic enslavement property described earlier. However, the mechanism through which this comes about for (2.3) is very different since only weak dissipation is present. Our analysis therefore sheds further light on the relation between the contractive property typically expected of this mechanism and unique ergodicity. Indeed, the proof of unique ergodicity adopts the approach of *asymptotic coupling* studied in [51], where it was successfully applied to a number of *strongly dissipative* systems such as the 2D NSE. In this approach, a coupling is designed so that one process *asymptotically synchronizes* with the original one. In order to satisfy the Novikov condition, this synchronization is only enforced on *large scales*, specifically the finitely-many scales that are directly injected with noise.

This result is only the second of such results for weakly dissipative systems like KdV, the first being [34] for the damped-driven nonlinear Schrödinger equation. However, we emphasize that our approach has the advantage of being both conceptually simpler and paradigmatic, as it applies just as well to the case treated in [34], and expectantly, to a number of other weakly dissipative, dispersive-type systems. Our analysis also yields regularity of the invariant measure as a byproduct. On the other hand, mixing rates to the invariant measure are currently unavailable and remains an ongoing investigation. Other related considerations such controllability of (2.3) and the "hypoelliptic" case, i.e., N is independent of γ, are important directions that we continue to push in.

3. A Feedback-Control Paradigm for Signal Synchronization

An approach to DA and downscaling (DS) that has been another focus of my work is the one proposed by Azouani, Olson, and Titi in [2]. Suppose that u is a solution to a physical model given by

$$\frac{du}{dt} = F(u), \tag{3.1}$$

except that the initial data u_0 has *not* been provided and is thus, *unknown*. Instead, consider the IVP:

$$\frac{dv}{dt} = F(v) - \mu I_h(v - u), \quad v(0) = v_0, \tag{3.2}$$

where v_0 is *any* initial condition, $h > 0$, $\mu = \mu(h) > 0$ is the "tuning parameter", and I_h is a finite-rank linear operator that interpolates observations to the phase space of (3.2). Typically, I_h represents projection onto finitely many nodal or modal values, where h quantifies observation density. One would then integrate (3.2) forward in time to obtain an approximation to the reference solution, u. Indeed, $-\mu I_h(v - u)$ serves to relax v towards u, but *only on large scales*. One can then show that this, in fact, suffices to synchronize v with u at *small scales as well* by exploiting the *asymptotic enslavement property* described in the previous section, whenever it exists.

Contributions. A third arm of my research program studies the feedback control paradigm (3.2) for a host of nonlinear PDEs in hydrodynamics in a variety of situations in DA and DS of dynamical systems, such as studying different forms of observations [5, 11, 67, 69], the strength of synchronization [7], scenarios of model error [42], and the problem of estimating unknown parameters [18, 86].

7. In [11, 67], we modify the feedback control system (3.2) to accommodate the more physical case of *time-averaged modal observables with a delay*, i.e., $\bar{I}_h = \frac{1}{\delta}\int_{t-2\delta}^{t-\delta} I_h$, where I_h is given by projection onto finitely many Fourier modes. Indeed, this is motivated by the fact that measurement devices used to collect data, e.g. wind velocity, temperature, is manifestly averaged in time. We are able to ensure the synchronization property *in spite of the delay* and with essentially the *same* number of modes as the case of instantaneous-in-time measurements, provided that the averaging window, δ, is sufficiently small. We overcome the difficulties introduced by the temporal non-locality in $\bar{I}_h$ by controlling the time-derivative at *large scales*, and establishing a non-local Gronwall inequality. We study this in the case of the Lorenz system and (1.2) when $\gamma > 0, \kappa > 1 = \beta$, with periodic boundary conditions, the latter of which requires us to establish new approximation inequalities involving fractional derivatives (see [69]).
8. In [42], we study a situation of model error in the context of the 3D Bousinessq equations for Rayleigh-Bénard Convection, which, in non-dimensionalized variables, is given by

$$\begin{gathered}\frac{1}{\mathrm{Pr}}\left[\partial_t u + (u\cdot\nabla)u\right] - \Delta u = -\nabla p + \mathrm{Ra}\,\mathbf{e}_3 T, \quad \nabla\cdot u = 0,\\ \partial_t T + u\cdot\nabla T - \Delta T = 0,\\ u|_{x_3=0} = u|_{x_3=1} = 0, \quad T|_{x_3=0} = 1,\ T|_{x_3=1} = 0, \quad u, T \text{ periodic in } x_1 \text{ and } x_2,\end{gathered} \tag{3.3}$$

over the domain $\Omega = [0, L]^2 \times [0, 1]$, where temperature, T, is advected along the fluid velocity, u, but due to buoyancy effects, can also drive the fluid motion. The Prandtl number, Pr, represents the relative strength of viscosity to thermal diffusivity, while the Rayleigh number, Ra, captures the strength of the buoyancy. On geological time scales, earth's mantle can be approximated as an incompressible fluid. When $\mathrm{Pr} = \infty$, (3.3) models mantle flow with the convective effects due to heating by earth's core. The situation of model error conceived in [42] is that of *temperature-only* observations corresponding to the large, but finite–Pr system, but that feedback control (3.2) is implemented through the infinite–Pr system. Although the issue of global regularity of (3.3) is open when $\mathrm{Pr} < \infty$, we exploit the property of *eventual regularization* [90] to establish synchronization up to an error depending on Pr, Ra and provide a battery of numerical simulations probing the relationship between $\mu, \mathrm{Pr}, \mathrm{Ra}$. We also explore sensitivity of the synchronization to the number of observables. We will continue this line of investigations in other geophysical situations when rotation or stratification effects are present.
9. In [18, 86], we study the problem of estimating unknown parameters of dynamical systems when knowledge of only a subset of state variables are known in the form of a continuous time series. We study this problem in the case of the Lorenz system and 2D NSE. The algorithm of interest is the one introduced in [17] for the 2D NSE based on the feedback-control paradigm (3.2) that proposes increasingly accurate values of the unknown viscositiy at judiciously chosen times. There, a sensitivity-type analysis is performed alongside a number of numerical tests that study the efficacy of the algorithm for the 2D NSE. A proof of convergence, however, remained open. This was supplied in [18] for the Lorenz system, and, in a forthcoming paper [86], for the 2D NSE. Both convergence proofs rely on a certain non-degeneracy condition that is to be satisfied as the time that parameter values are to be updated. This condition is numerically probed in the case of the Lorenz system, where we observe that the non-degeneracy condition to hold in favorable parameter regimes. These results appear are the first of this kind for parameter estimation in nonlinear PDEs and open the door to rigorous proofs of convergence for other systems. Ultimately, we would like to study more realistic cases that incorporate model or observational noise as well.

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