Research Statement

My research interests are rooted in the analysis of partial differential equations (PDEs), particularly of hydrodynamic or geophysical equations, such as the incompressible Navier-Stokes or Euler equations, the surface quasi-geostrophic (SQG) equations, or various related equations, such as chemotaxis equations that incorporate the effects from interaction of an organism with an ambient incompressible fluid, and dispersive equations that incorporate weak damping mechanisms. A large source of inspiration derives from the mathematics of turbulence through the characterization of small length scales, manifestations of finite-dimensionality in the long-time regime, as well as its applications, for instance, to dissipative dynamical systems, data assimilation (DA), or parameter estimation. In order treat these various considerations, my work often employs tools and techniques from harmonic analysis, elliptic, parabolic, and hyperbolic equation theory, infinite-dimensional analysis, approximation theory, and control theory. Due to the interdisciplinary nature of my work, it is also accompanied and at times driven by computational efforts. Generally speaking, my research moves along three interrelated directions: (I) Well-posedness and Regularity, (II) Long-time Behavior: Deterministic and Statistical, (III) Applications to DA and Parameter Estimation. In what follows, I describe selected works in each direction, identify how perspectives from turbulence enter some of them, and mention current investigations and future ones that emanate from them.

1. Well-posedness and Regularity

Well-posedness of the Cauchy initial value problem (IVP) is a fundamental issue in the study of evolutionary equations as it asserts the existence, uniqueness, and continuity with respect to initial data of solutions. In hydrodynamics, this most basic form of validation for a physical model has yet to be fully settled for the equations of motion for a three-dimensional (3D), incompressible viscous fluid, that is, the 3D Navier-Stokes equations (NSE). In $\mathbb{R}^d$, $d = 2, 3$, one has

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0, \quad u(0, x) = u_0(x), \tag{1.1}$$

where $u = u(t, x)$ denotes the velocity vector field of the fluid, $\nabla\cdot u = 0$ expresses its incompressibility, p is the scalar pressure field, u_0 is the initial velocity field, ν is the kinematic viscosity, and (1.1) is appropriately supplemented wtih decay at infinity. The case $\nu = 0$ corresponds to the Euler equations and is referred to as the *inviscid case*. Although global-in-time existence and uniqueness of weak solutions is known in 2D [84], and in 3D, global existence of weak solutions satisfying an energy inequality [62, 84] and local existence *and* uniqueness of strong solutions to (1.1) is known [45, 54, 65, 73, 77], the problem of whether such weak solutions are unique when $\nu > 0$ or if singularities can develop for strong solutions in finite time have been outstanding open problems since the equations were conceived in the 19th century. The latter is known as the *global regularity problem* for the 3D NSE and is listed by the Clay Mathematics Institute alongside the Riemann Hypothesis, the Hodge Conjecture, and several others as one of the great unsolved problems in mathematics. The study of the well-posedness of (1.1) continues to be a rich vein of research and source of mathematical problems, as evidenced by recent breakthroughs in its understanding: non-uniqueness of weak solutions to 3D Euler [19, 39, 64], of 3D NSE [20, 28], loss of continuity with respect to initial data for 3D Euler [17, 47, 48] and 3D NSE [18], and finite-time blow-up in the Hölder class to 3D Euler [46].

Contributions. One arm of my research is dedicated to shedding light on well-posedness and regularity properties of hydrodynamic equations and related systems. These models share some structural similarities to (1.1), but typically enjoy a reduced dimensionality or other regularizing mechanisms. One important family of such models are given by the *generalized* SQG (gSQG) equations (see [24–26]):

$$\partial_t\theta + \gamma\Lambda^\kappa\theta + (u\cdot\nabla)\theta = 0, \quad u = -\nabla^\perp\Lambda^{\beta-2}\theta, \quad \theta(0, x) = \theta_0(x), \tag{1.2}$$

where $\kappa \in (0, 1)$, $\beta \in [0, 2)$, $\gamma \geq 0$, and $\theta = \theta(t, x)$ is a scalar. We refer to $\gamma = 0$ as the *inviscid* gSQG. In the inviscid case, (1.2) interpolates between 2D Euler ($\beta = 0$) and the SQG ($\beta = 1$), then extrapolates beyond with an increasingly singular constitutive law. A great deal of effort has gone towards the understanding of the well-posedness of (1.2) since its introduction to the mathematical community by Constantin, Majda, and Tabak [31], especially due to its structural analogy to (1.1) in 3D when $\beta = 1$ and either $\kappa = 1$ (to NSE) or $\gamma = 0$ (to Euler) (see, for instance, [21, 32–36, 66, 74–76, 91]).

In a series of works with collaborators [13, 67, 68, 82], we probe the relation between the dissipation and constitutive law in terms of well-posedness. The choice of functional setting is one of *borderline regularity* in the sense that it is the lowest level of regularity for which one can expect well-posedness; it is typically characterized by the scaling symmetry of the equation. At or below the level of *critical regularity* well-posedness can either fail or hold in various ways: non-existence, non-uniqueness, or loss of continuity with respect to initial data. On the other hand, due to the non-negative definite nature of the dissipation, smoothing effects also arise; the extent to which this phenomenon still holds at borderline regularity constitutes one of the motivations for these works. For this, it is convenient to consider the following generalization to (1.2):

$$\partial_t\theta + \gamma m(D)\theta + (u\cdot\nabla)\theta = 0, \quad u = -\nabla^\perp a(D)\theta, \tag{1.3}$$

where $\mathcal{F}(m(D)\theta)(\xi) = m(\xi)(\mathcal{F}\theta)(\xi)$, and $m(D), a(D)$ are non-negative, radial multiplier operators. The main achievements are two-fold. Firstly, we extend results in L^2–based settings to L^p–based ones [13], which crucially exploits insights in bilinear multiplier operator theory. Secondly, in [67,68,82], we treat the most singular regimes of the constitutive law, which had hitherto been left open in borderline regularity settings apparently due to lack of a proper way to approximate solutions that sufficiently respects the underlying the commutator structure of the equations; we identify such an approximation scheme to provide a comprehensive study of well-posedness.

1. ($m(D) = \Lambda^\kappa, a(D) = \Lambda^{-1}$, $\kappa \in (0,1)$) In this case, (1.3) possesses a scaling symmetry, $\theta_\lambda = \lambda^{\kappa-1}\theta(\lambda^\kappa t, \lambda x)$. We study the supercritical regime, $\kappa < 1$ in [13] and show that (1.3) has local existence and uniqueness for large data in the scaling-critical Besov space, $\dot{B}^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)$, for $2 \leq p < \infty$ and $1 \leq q \leq \infty$, and global existence and uniqueness for small data. In spite of supercriticality, we show that the maximal spatial regularity arising from the parabolic operator $L = \partial_t + \gamma\Lambda^\kappa$ is conferred instantaneously. In doing so, we extend the technique of Gevrey norms developed in [52] from L^2–based Sobolev spaces to L^p–based Besov spaces. Due to supercriticality, new commutator estimates were needed, particularly for estimates in Besov–based Gevrey classes. The novelty was to view the commutator as a bilinear multiplier operator and verify a Marcinkiewicz-type condition. Owing to various localizations arising from working in the Besov space setting, we show that this condition is sufficient to obtain $L^p \times L^q \to L^r$–type bounds, even though it is well-known that such bilinear operators do not satisfy *any* such bounds in general [57]. Our result also complements ones in the critical space setting previous to this for the 3D NSE [3,55] and 2D subcritical SQG [43].
2. ($m(D) = \Lambda^\kappa, a(D) = \Lambda^{\beta-2}$, $\beta \in (0,1)$) Again, (1.3) has a scaling symmetry, $\theta_\lambda = \lambda^{\kappa-\beta}\theta(\lambda^\kappa t, \lambda x)$ and in [68] we establish results analogous to those in [13], but in the setting of scaling-critical Sobolev spaces, $\dot{H}^{\beta+1-\kappa}(\mathbb{R}^2)$. This extends the results in [9, 90] to the most singular range of the gSQG family. Due to the more singular nature of the constitutive law, the commutator structure of (1.3) is exploited in a more nuanced manner. The commutators identified in [24,63] allow one to carry out an apriori analysis. However, the scaling-critical setting still requires one to *construct* the solution. Since stability–type estimates for (1.3) at critical regularity are not known, one *cannot* simply carry out a density argument with smooth initial data. We overcome this by proposing a linear conservation law that approximates the inviscid part of the system such that the divergence of its flux collapses to the original advective term in the limit. This approximation scheme suitably preserves the underlying commutator structure and ultimately allows for the construction of solutions to (1.3) at critical regularity. Our analysis identifies an *additional* structural criticality when $\kappa = \beta - 1$; above this line, (1.3) exhibits a "strongly" quasilinear structure since its "coefficients" have an order that strictly exceeds the order of the linear part; below it, the approximation procedure is classical. We would now like to extend these results to the L^p setting and carry out the critical space program of the 3D NSE to the 2D gSQG.
3. ($m(D) = 0, a(D) = \log^\mu(e-\Delta)\Lambda^{\beta-2}$, $\mu \in (1/2, \infty)$, $\beta \in (1,2)$) Then (1.3) is inviscid with constitutive law *mildly* regularized by a power of a log-laplacian. In [67], we show that (1.3) is locally well-posed in the borderline Sobolev space, $H^{\beta+1}(\mathbb{R}^2)$, that is, one has existence, uniqueness, as well as continuity with respect to initial data. This extends the work [27], which establishes local well-posedness for $\beta \in [0,1]$. The most important distinction between $\beta \in [0,1]$ and $\beta \in (1,2)$ does *not* arise in the apriori analysis, but rather in proving stability. Indeed, in [27] continuity with respect to initial data follows by classical means (see [85]). Such an approach fails in range $\beta \in (1,2)$ and it becomes crucial to exploit the commutator structure of (1.3). We do this by modifying a splitting technique of Kato [72] for symmetric hyperbolic systems that preserves the more nuanced commutator structure of the equation. We would like to explore other forms of mild regularization of systems and attempt to characterize the phenomenon of borderline well-posedness.

2. Long-time Behavior: Deterministic and Statistical

According to the Kolmogorov 1941 theory of 3D turbulence [78,79], energy cascades from large scales to small scales through a nonlinear mechanism. This cascade should then extend down to the so-called *dissipation length scale*, which indicates the scale at which nonlinear energy transfer is in an exact balance with viscous dissipation. At this scale, the energy spectrum experiences an exponential drop-off. Kolmogorov posited that this length scale, ℓ_{Kol}, is uniquely determined by the viscosity and the average rate of energy dissipation in the flow. In particular, ℓ_{Kol} represents the smallest relevant length scale in the system. From this, Landau and Lifshitz [83] defined the number of degrees of freedom, N_{LL}, to be the total number of eddies of this size that saturate the size of the domain, ℓ_{dom}^3 (if the domain is a box), so that $N_{LL} = (\ell_{dom}/\ell_{Kol})^3$. Kolmogorov's phenomenology of 3D turbulence, as well as its 2D counterpart by Batchelor-Kraichnan [6,80], have provided a rich source of mathematical investigations, particularly through providing rigorous confirmation of these predictions directly through the equations of motions themselves [4,5,7,8,29,30,37,38,42].

Contributions. A second arm of my research is therefore focused on either studying these issues directly or else attempting to capture various features of the long-time behavior of solutions to hydrodynamic and related systems that are either deterministically or stochastically perturbed. Indeed, "turbulence" in a general sense is a phenomenon where the scales at which energy is injected and those where energy is dissipated are separated far away and realizable only in a "permanent" regime often characterized by passage to the infinite-time limit. It is therefore natural and fundamental to the study of turbulence to consider various setups for the energy injection and dissipation. These variations can be realized by deterministic or stochastic perturbations of the equation or through considering different forms of dissipation, either through viscous dissipation or various forms of damping. These studies are performed in a series of works dating back to my Ph.D. thesis [11] and continuing in [56,60,61,70]. I also study forms of stability in other systems arising in chemotaxis in [1,87,88,93].

In the following, I describe three representative works that address these issues. The main achievements in this direction are in 1) obtaining refined estimates on the dissipation length scale of Kolmogorov for 3D turbulent flows and Kraichnan for 2D turbulent flows through a unified framework for estimating the real analyticity radius of solutions [11], 2) studying a particular manifestation of finite-dimensionality of dynamics in a geophysical scenario, where dissipation is given by a non-local operator, in order to imbed the dynamics of (1.2) into that of an ODE [70], and 3) verifying the ergodic hypothesis for a *weakly-damped*, stochastically forced system by greatly expanding an approach developed originally developed for strongly-dissipative systems [56].

4. In [11], the Gevrey-norm technique of Foias and Temam [52] is adapted to a Wiener algebra framework, i.e., L^1–based, to obtain refined estimates on the real analyticity radius, ℓ_a, for the (1.1) over a periodic domain in *any* spatial dimension. Estimates on ℓ_a yield estimates on the corresponding *dissipation wave-number*, $\kappa_a = \ell_a^{-1}$. This subsequently indicates where the energy spectrum experiences *exponential decay*, which ultimately leads to an upper bound estimate on N_{LL}. This point of view was developed in several works [11,14,15,41,52,58,59,81]. In [11], we prove
$$\ell_a \gtrsim \ell_{Kol}^4 \text{ (in 3D)} \quad \text{and} \quad \ell_a \gtrsim \ell_{Kr}^2 \text{ (in 2D)}, \tag{2.1}$$
for turbulent flows that satisfy (1.1) in a periodic domain, where ℓ_{Kr} refers to the Kraichnan dissipation length scale, which is the 2D analog of the Kolmogorov dissipation length scale represented by ℓ_{Kol}. The best-to-date estimates for ℓ_{Kr} in the periodic setting were obtained in [81] resorting to complex-analytic techniques. While the work of [41] established the best-to-date estimates for ℓ_{Kol} in the same setting, but with an L^2–based Gevrey-norm approach in the same setting. These estimates are captured by (2.1). Hence, our result unifies the results of [41] in 3D and [81] in 2D *under a single framework*. Moreover, our method exposes *a new path* to lowering the above exponents closer to 1, namely, by improving higher-order estimates of the flow *in the long-time average*, which would appeal to a statistical framework of turbulence. This makes revisiting the *stochastically forced case*, as originally studied in [89], a compelling future venture.
5. In the work [70], we establish the existence of a *determining form* (DF) for the subcritical SQG equation, i.e., (1.2), $\kappa \in (1,2)$ and $\beta = 1$, that is induced by a feedback control system (see (3.2)). A DF is an *ODE*, albeit in a Banach space of trajectories, that subsumes the global attractor of the original equation in a certain way. Motivated by finite dimensionality of the dynamics of 2D NSE manifested through "asymptotic enslavement of scales" [51], the notion of DFs was introduced in [50,53]. A stronger formulation of the notion of enslavement of scales is embodied in the existence of an *inertial form*, (IF) which is a *finite-dimensional*

system of ODEs governing the large scale evolution that is *fully decoupled* from the small scale evolution. However, the existence of an IF for (1.1) in 2D has been an outstanding open problem.

In [70], we specifically show that there exist Banach spaces, X, Y, and a map $W : X \to Y$, which is the solution operator of (3.2) corresponding to a "reference solution," $u \in X$, such that $I_hW : B^\rho_X(0) \to Y$ is *Lipschitz*, for some ball of radius $\rho > 0$, centered at 0 in X, where I_h is given by (smooth) projection onto Fourier modes $|k| \leq 2^{1/h}$. It then follows from [53], for instance, that the equation given by

$$\frac{dv(\cdot)}{d\tau}(\tau) = -\|v(\cdot)(\tau) - I_hW(v(\cdot)(\tau))\|^2_X(v(\cdot)(\tau) - I_h\theta^*), \quad v(0) = v_0 \in \mathcal{B}^\rho_X(0). \tag{2.2}$$

is defined by a right-hand that is Lipschitz and hence, defines an *ODE* in X, where θ^* is a given steady state of (1.2). Our proof that (2.2) defines an ODE hinges on obtaining uniform estimates in L^∞ for solutions to (3.2) *independent* of the number of modes $m \sim 2^{1/h}$. Indeed, the L^p–maximum principle one is able to derive for (3.2) *depends on the number of modes*, which precludes one from using these bounds to establish the Lipschitz property. Nevertheless, by appealing to harmonic analysis tools, we show that one can adapt De Giorgi techniques [21] to deduce uniform L^∞–bounds independent of m. Interestingly, the mechanism of asymptotic enslavement is also a crucial ingredient to establishing uniqueness of invariant measures for the 2D NSE. We would therefore like the study the extent to which the identification of a DF can provide any computational or theoretical gain in understanding invariant measures of stochastically perturbed systems.

6. In [56], existence and uniqueness of invariant probability measures is proved for the damped-driven Korteweg-de Vries (KdV) equation

$$du + (\gamma u + u_{xxx} + uu_x)dt = fdt + \sigma dW, \tag{2.3}$$

where a large, but finite number $N = N(\gamma) \gg 1$ of Fourier modes are stochastically forced. Here, u represents an amplitude, $\gamma > 0$ captures damping effects, f is an external, time-independent deterministic forcing, and $\sigma W = \sum_{j=1}^N \sigma_j W_j$ is a Wiener process such that $\sum_{j=1}^N \|\sigma_j\|^2_{H^2} < \infty$ and each $W_j = W_j(t)$ is a 1D standard Brownian motion. A typical assumption made in studying of turbulence is to *assume* ergodicity of its dynamics. This assumption asserts that long-time averages can be equated with ensemble averages, that is, averages over all possible states of the system, and is often referred to as the *ergodic hypothesis*. Since ergodic invariant measures form the extremal points of the set of invariant probability measures, uniqueness of such measures guarantees its ergodicity. In [56], we thus verify the ergodic hypothesis for (2.3).

The propagation of the noise through the system, which is crucial to establishing uniqueness, is well-understood for the 2D NSE [44] and arises due to the balance achieved between heat dissipation and nonlinear effects that manifests in the asymptotic enslavement property described earlier. However, the mechanism through which this arises for (2.3) is very different since the form of dissipation is categorically *weaker* than the laplacian. Our analysis thus sheds light on the relation between the contractivity property implied by asymptotic enslavement and that of unique ergodicity. Our proof adopts an *asymptotic coupling* approach developed which had been successfully developed for a number of *strongly dissipative* systems such as the 2D NSE. In this approach, a coupling is designed so that one process *asymptotically synchronizes* with the original one. Before [56], it was not known whether this approach could be applied to systems such as (2.3).

This result is only the second of such results for weakly dissipative systems like KdV, the first being [40] for the damped-driven nonlinear Schrödinger (NLS) equation. However, the approach in [40] constructed a highly technical *exact coupling*. In contrast, we effectively observe that by relaxing the form of coupling, one obtains approach that is both conceptually simpler and paradigmatic. As a byproduct of our analysis, we also obtain regularity of the invariant measure, which in the literature is often proved as a result on its own. On the other hand, mixing rates for the Markov semigroup associated to (2.3) are currently not known; this remains an ongoing investigation. Other related considerations such as controllability of (2.3) and the "hypoelliptic" case, i.e., N is independent of γ, raise important questions that we continue to pursue.

3. A Feedback-Control Paradigm for Synchronization

An effective approach for studying problems in DA for partial differential equations was developed by Azounai, Olson, and Titi in [2]: Suppose that u represents a physical phenomenon governed by the following system

$$\frac{du}{dt} = F(u), \tag{3.1}$$

except that the initial data u_0 has *not* been provided and is thus, *unknown*. One instead considers the system

$$\frac{dv}{dt} = F(v) - \mu I_h(v) + \mu I_h(u), \quad v(0) = v_0, \tag{3.2}$$

where v_0 is *any* initial condition, $\mu = \mu(h) > 0$ is a "tuning parameter," and $I_h(u)$ represents the collected observations. Typically, I_h represents linear projection onto finitely many nodal values or spectral modes and h quantifies observational density; it suitably interpolates observations into the phase space of (3.1) and facilitates its insertion into the model. Then by integrating (3.2) forward in time, one obtains an approximation, v, to the reference solution, u, that is, v *synchronizes* with u.

Contributions. The main results achieved in this component of my research are 1) extension of the feedback-control approach to accommodate increasingly physical forms of observation [10, 16, 69, 71], 2) proof of that synchronization is in fact typically achieved in all higher-order topologies [12], 3) rigorously study scenarios of model error [49], and 4) develop and provide convergence analysis of algorithms for parameter estimation in nonlinear systems [23, 86]. We describe three representative results below.

7. In [16, 69], we modify the feedback control system (3.2) to accommodate the more physical case of *time-averaged modal observables with a delay*, i.e., $\bar{I}_h = \frac{1}{\delta}\int_{t-2\delta}^{t-\delta} I_h$, where I_h is given by projection onto finitely many Fourier modes. Indeed, this is motivated by the fact that measurement devices used to collect data, e.g. wind velocity, temperature, is manifestly averaged in time. We prove synchronization occurs *in spite of the delay* and with essentially the *same* number of modes as the case of instantaneous-in-time measurements, provided the averaging window, δ, is sufficiently small. We overcome difficulties introduced by the temporal non-locality in $\bar{I}_h$ by controlling the time-derivative at *large scales*, and establishing a *non-local Gronwall inequality*. We study this in the case of the Lorenz system [16] and (1.2) when $\gamma > 0, \kappa > 1 = \beta$ [69]; the latter case requires us to establish new approximation inequalities involving fractional derivatives (see [71]). We would now like to a system with memory, which may interact with the delay in an interesting way.
8. In [49], we study a situation of model error in the context of the 3D Bousinessq equations for Rayleigh-Bénard convection, which, in non-dimensionalized variables, is given by

$$\frac{1}{\mathrm{Pr}}\left[\partial_t u + (u\cdot\nabla)u\right] - \Delta u = -\nabla p + \mathrm{Ra}\,\mathbf{e}_3 T, \quad \nabla\cdot u = 0, \quad \partial_t T + u\cdot\nabla T - \Delta T = 0, \tag{3.3}$$

over the domain $\Omega = [0, L]^2 \times [0, 1]$, where T denotes temperature and forces the fluid with velocity u through buoyancy effects by heating at the bottom boundary and cooling at the top, with Dirichlet boundary conditions for u in vertical direction. The Prandtl number, Pr, captures the relative strength of viscosity to thermal diffusivity, while the Rayleigh number, Ra, captures the strength of the buoyancy. On geological time scales, the earth's mantle can be viewed as an incompressible fluid. When $\mathrm{Pr} = \infty$, (3.3) models mantle flow with convective effects due to heating by the earth's core. The situation of model error conceived in [49] is that of *temperature-only* observations corresponding to the large, but finite–Pr system, but where (3.2) is implemented through the $\mathrm{Pr} = \infty$ system. Although global regularity of (3.3) is not known when $\mathrm{Pr} < \infty$, we exploit the property of *eventual regularization* [92] to establish synchronization up to an error depending on Pr, Ra. A battery of numerical experiments probing the relationship between $\mu, \mathrm{Pr}, \mathrm{Ra}$ is also carried out. This line of investigation in other geophysical situations when rotation or stratification effects are present are currently being explored.
9. In [23, 86], we study the problem of estimating unknown parameters of nonlinear dynamical systems when knowledge of only a subset of state variables is given as time series. The algorithm of interest is the one introduced in [22] for the 2D NSE based on the feedback-control paradigm (3.2) that proposes increasingly accurate values of the unknown viscosity at judiciously chosen times. Although a sensitivity-type analysis was performed alongside a number of numerical tests that study the efficacy of the algorithm for the 2D NSE, a proof of convergence remained open. This was finally achieved in [23] and [86]. The proofs rely on a non-degeneracy (ND) condition to be satisfied at the times when updates to parameter value are made. This condition is numerically probed in the case of the Lorenz system, where we observe that the ND condition holds in favorable parameter regimes. These results appear to be the first of their kind for parameter estimation in nonlinear equations and opens the door to rigorous proofs of convergence for other systems.

References

[1] P.F. Aguilera, V.R. Martinez, and K. Zhao. A pde model for chemotaxis with logarithmic sensitivity and logistic growth. *Contemp. Math. Appl., Monogr. Expo. Lect. Notes*, pages 1–23, 2020.

[2] A Azouani, E.J. Olson, and E.S. Titi. Continuous data assimilation using general interpolant observables. *J. Nonlinear Sci.*, 24(2):277–304, 2014.

[3] H. Bae. Existence and analyticity of Lei-Lin solution to the Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 143(7):2887–2892, 2015.

[4] N. Balci, C. Foias, and M.S. Jolly. 2-D turbulence for forcing in all scales. *J. Math. Pures Appl. (9)*, 94(1):1–32, 2010.

[5] N. Balci, C. Foias, M.S. Jolly, and R. Rosa. On universal relations in 2-D turbulence. *Discrete Contin. Dyn. Syst.*, 27(4):1327–1351, 2010.

[6] G.K. Batchelor. Computation of the energy spectrum in homogeneous two-dimensional turbulence. *Physics of fluids*, 12(Suppl. II):233–239, 1969.

[7] J. Bedrossian, M Coti-Zelati, S. Punshon-Smith, and F. Weber. A sufficient condition for the Kolmogorov 4/5 law for stationary martingale solutions to the 3D Navier-Stokes equations. *Comm. Math. Phys.*, 367(3):1045–1075, 2019.

[8] J. Bedrossian, M. Coti-Zelati, S. Punshon-Smith, and F. Weber. Sufficient conditions for dual cascade flux laws in the stochastic 2d Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 237(1):103–145, 2020.

[9] A. Biswas. Gevrey regularity for the supercritical quasi-geostrophic equation. *J. Differential Equations*, 257(1):1753–1772, 2014.

[10] A. Biswas, K. Brown, and V.R. Martinez. Higher-order synchronization of a nudging-based algorithm for data assimilation for the 2D NSE: a refined paradigm for global interpolant observables. *Ann. Appl. Math.*, 38(3):296–355, 2022.

[11] A. Biswas, M.S. Jolly, V.R. Martinez, and E.S. Titi. Dissipation Length Scale Estimates for Turbulent Flows: A Wiener Algebra Approach. *J. Nonlinear Sci.*, 24:441–471, 2014.

[12] A. Biswas and V.R. Martinez. Higher-order synchronization for a data assimilation algorithm for the 2d Navier-Stokes equations. *Nonlinear Anal. Real World Appl.*, 35(1):132–157, 2017.

[13] A. Biswas, V.R. Martinez, and P. Silva. On Gevrey regularity of the supercritical SQG equation in critical Besov spaces. *J. Funct. Anal.*, 269(10):3083–3119, 2015.

[14] A. Biswas and D. Swanson. Gevrey regularity of solutions to the 3-D Navier-Stokes equations with weighted l_p initial data. *Indiana Univ. Math. J.*, 56(3):1157–1188, 2007.

[15] A. Biswas and D. Swanson. Gevrey regularity of solutions to the 3D Navier-Stokes equations. In *Fluids and waves*, volume 440 of *Contemp. Math.*, pages 83–90. Amer. Math. Soc., Providence, RI, 2007.

[16] J. Blocher, V.R. Martinez, and E. Olson. Data assimilation using noisy time-averaged measurements. *Phys. D*, 376/377:49–59, 2018.

[17] J. Bourgain and D. Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.*, 201(1):97–157, 2015.

[18] J. Bourgain and N. Pavlović. Ill-posedness of the Navier-Stokes equations in a critical space in 3D. *J. Funct. Anal.*, 255(9):2233–2247, 2008.

[19] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Jr. Anomalous dissipation for 1/5-Hölder Euler flows. *Ann. of Math. (2)*, 182(1):127–172, 2015.

[20] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math. (2)*, 189(1):101–144, 2019.

[21] L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math.*, 171(3):1903–1930, 2010.

[22] E. Carlson, J. Hudson, and A. Larios. Parameter recovery for the 2 dimensional Navier-Stokes equations via continuous data assimilation. *SIAM J. Sci. Comput.*, 42(1):A250–A270, 2020.

[23] E. Carlson, J. Hudson, A. Larios, V.R. Martinez, E. Ng, and J. Whitehead. Dynamically learning the parameters of a chaotic system using partial observations. *Discrete Contin. Dyn. Syst.*, pages 3809–2839, March 2022.

[24] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and J. Wu. Generalized surface quasi-geostrophic equations with singular velocities. *Comm. Pure Appl. Math.*, 65(8):1037–1066, 2012.

[25] D. Chae, P. Constantin, and J. Wu. Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations. *Arch. Ration. Mech. Anal.*, 202(1):35–62, 2011.

[26] D. Chae, P. Constantin, and J. Wu. Dissipative models generalizing the 2D Navier-Stokes and surface quasi-geostrophic equations. *Indiana Univ. Math. J.*, 61(5):1997–2018, 2012.

[27] D. Chae and J. Wu. Logarithmically regularized inviscid models in borderline Sobolev spaces. *J. Math. Phys.*, 53(11):115601, 15, 2012.

[28] M. Colombo, C. De Lellis, and L. De Rosa. Ill-posedness of Leray solutions for the hypodissipative Navier-Stokes equations. *Comm. Math. Phys.*, 362(2):659–688, 2018.

[29] P. Constantin, C. Foias, B. Nicolaenko, and R. Temam. *Integral manifolds and inertial manifolds for dissipative partial differential equations*, volume 70 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[30] P. Constantin, C. Foias, and R. Temam. On the dimension of the attractors in two-dimensional turbulence. *Phys. D*, 30(3):284–296, 1988.

[31] P. Constantin, A.J. Majda, and E. Tabak. Formation of strong fronts in the 2d quasigeostrophic thermal active scalar. *Nonlinearity*, 7:1495–1533, 1994.

[32] P. Constantin and V. Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geom. Funct. Anal.*, 22:1289–1321, 2012.

[33] A. Córdoba and D. Córdoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249(3):511–528, 2004.

[34] A. Córdoba, D. Córdoba, and F. Gancedo. Uniqueness for SQG patch solutions. *Trans. Amer. Math. Soc. Ser. B*, 5:1–31, 2018.

[35] D. Córdoba, J. Gómez-Serrano, and A.D. Ionescu. Global solutions for the generalized SQG patch equation. *Arch. Ration. Mech. Anal.*, 233(3):1211–1251, 2019.

[36] D. Córdoba and L. Martínez-Zoroa. Non existence and strong ill-posedness in c^k and sobolev spaces for sqg. *arXiv preprint arXiv:2107.07463*, pages 1–55, July 19, 2021.

[37] R. Dascaliuc, C. Foias, and M. S. Jolly. Relations between energy and enstrophy on the global attractor of the 2-D Navier-Stokes equations. *J. Dynam. Differential Equations*, 17(4):643–736, 2005.

[38] R. Dascaliuc, C. Foias, and M.S. Jolly. Some specific mathematical constraints on 2D turbulence. *Phys. D*, 237(23):3020–3029, 2008.

[39] Camillo De Lellis and László Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.

[40] Arnaud Debussche and Cyril Odasso. Ergodicity for a weakly damped stochastic non-linear schrödinger equation. *J. Evol. Equ.*, 5(3):317–356, 2005.

[41] C. Doering and E.S. Titi. Exponential decay rate of the power spectrum for solutions of the Navier-Stokes equations. *Phys. Fluids*, 7(6):1384–1390, 1995.

[42] C.R. Doering and C. Foias. Energy dissipation in body-forced turbulence. *J. Fluid Mech.*, 467:289–306, 2002.

[43] H. Dong and D. Li. Spatial analyticity of the solutions to the subcritical dissipative quasi-geostrophic equations. *Arch. Ration. Mech. Anal.*, 189(1):131–158, 2008.

[44] W. E, J. C. Mattingly, and Y. G.. Sinai. Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Comm. Math. Phys.*, 224(1):83–106, 2001. Dedicated to Joel L. Lebowitz.

[45] D.G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.

[46] T.M. Elgindi. Finite-time Singularity Formation for $C^{1,\alpha}$ Solutions to the Incompressible Euler Equations on $\mathbb{R}^3$. *arXiv preprint arXiv:1904.04795*, 2019.

[47] T.M. Elgindi and I. Jeong. Ill-posedness for the incompressible Euler equations in critical Sobolev spaces. *Ann. PDE*, 3(1):Paper No. 7, 19, 2017.

[48] T.M. Elgindi and N. Masmoudi. L^∞ ill-posedness for a class of equations arising in hydrodynamics. *Arch. Ration. Mech. Anal.*, 235(3):1979–2025, 2020.

[49] A. Farhat, N. E. Glatt-Holtz, V. R. Martinez, S. A. McQuarrie, and J. P. Whitehead. Data assimilation in large Prandtl Rayleigh-Bénard convection from thermal measurements. *SIAM J. Appl. Dyn. Syst.*, 19(1):510–540, 2020.

[50] C. Foias, M.S. Jolly, R. Kravchenko, and E.S. Titi. A determining form for the two-dimensional Navier-Stokes equations: the Fourier modes case. *J. Math. Phys.*, 53(11):115623, 30, 2012.

[51] C. Foiaş and G. Prodi. Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2. *Rend. Sem. Mat. Univ. Padova*, 39:1–34, 1967.

[52] C. Foias and R. Temam. Gevrey Class Regularity for the Solutions of the Navier-Stokes Equations. *J. Funct. Anal.*, 87:359–369, 1989.

[53] K. Foyash, M. S. Dzholli, R. Kravchenko, and È. S. Titi. A unified approach to the construction of defining forms for a two-dimensional system of Navier-Stokes equations: the case of general interpolating operators. *Uspekhi Mat. Nauk*, 69(2(416)):177–200, 2014.

[54] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16:269–315, 1964.

[55] P. Germain, N. Pavlović, and G. Staffilani. Regularity of solutions to the Navier-Stokes equations evolving from small data in BMO^{-1}. *Int. Math. Res. Not. IMRN*, (21):Art. ID rnm087, 35, 2007.

[56] N. Glatt-Holtz, V.R. Martinez, and G. Richards. On the long-time statistical behavior of smooth solutions of the weakly damped, stochastically-driven KdV equation. *arXiv:2103.12942v1 (submitted)*, pages 1–70, Mar 24, 2021.

[57] Loukas Grafakos and Nigel J. Kalton. The Marcinkiewicz multiplier condition for bilinear operators. *Studia Math.*, 146(2):115–156, 2001.

[58] Z. Grujić and I. Kukavica. Space analyticity for the Navier-Stokes and Related Equations with Initial Data in l^p. *J. Funct. Anal.*, 152:447–466, 1998.

[59] W. Henshaw, H.O. Kreiss, and L.G. Reyna. Long time dynamics of forced critical sqg. *Arch. Rational Mech. Anal.*, 12:21–44, 1990.

[60] L.T. Hoang and V.R. Martinez. Asymptotic expansion in Gevrey spaces for solutions of Navier-Stokes equations. *Asymptot. Anal.*, 104(3-4):167–190, 2017.

[61] L.T. Hoang and V.R. Martinez. Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces. *J. Math. Anal. Appl.*, 462(1):84–113, 2018.

[62] E. Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[63] W. Hu, I. Kukavica, and M. Ziane. Sur l'existence locale pour une équation de scalaires actifs. *C. R. Math. Acad. Sci. Paris*, 353(3):241–245, 2015.

[64] P. Isett. A proof of Onsager's conjecture. *Annals of Mathematics*, 188(3):871–963, 2018.

[65] S. Itô. The existence and the uniqueness of regular solution of non-stationary Navier-Stokes equation. *J. Fac. Sci. Univ. Tokyo Sect. I*, 9:103–140 (1961), 1961.

[66] I.-J. Jeong and J. Kim. Strong ill-posedness for sqg in critical sobolev spaces. *arXiv preprint arxiv:2107.07739*, pages 1–35, July 16, 2021.

[67] M.S. Jolly, A. Kumar, and V.R. Martinez. On local well-posedness of logarithmic inviscid regularizations of generalized SQG equations in borderline Sobolev spaces. *Commun. Pure Appl. Anal.*, pages 1–20, 2021.

[68] M.S. Jolly, A. Kumar, and V.R. Martinez. On the existence, uniqueness, and smoothing of solutions to the generalized SQG equations in critical Sobolev spaces. *Commun. Math. Phys.*, pages 1–46, 2021. DOI:10.1007/s00220-021-04124-9.

[69] M.S. Jolly, V.R. Martinez, E.J. Olson, and E.S. Titi. Continuous data assimilation with blurred-in-time measurements of the surface quasi-geostrophic equation. *Chin. Ann. Math. Ser. B*, 40(5):721–764, 2019.

[70] M.S. Jolly, V.R. Martinez, S. Sadigov, and E.S. Titi. A Determining Form for the Subcritical Surface Quasi-Geostrophic Equation. *J. Dyn. Differ. Equations*, pages 1–38, 2018.

[71] M.S. Jolly, V.R. Martinez, and E.S. Titi. A data assimilation algorithm for the 2d subcritical surface quasi-geostrophic equation. *Adv. Nonlinear Stud.*, 35:167–192, 2017.

[72] T. Kato. Quasilinear equations of evolution in nonreflexive Banach spaces. In *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, volume 81 of *North-Holland Math. Stud.*, pages 61–76. North-Holland, Amsterdam, 1983.

[73] T. Kato and H. Fujita. On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova*, 32:243–260, 1962.

[74] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167:445–453, 2007.

[75] A. Kiselev, L. Ryzhik, Y. Yao, and A. Zlatoš. Finite time singularity for the modified SQG patch equation. *Ann. of Math. (2)*, 184(3):909–948, 2016.

[76] A. Kiselev, Y. Yao, and A. Zlatoš. Local regularity for the modified SQG patch equation. *Comm. Pure Appl. Math.*, 70(7):1253–1315, 2017.

[77] A. A. Kiselev and O. A. Ladyženskaya. On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 21:655–680, 1957.

[78] A. N. Kolmogorov. On degeneration of isotropic turbulence in an incompressible viscous liquid. In *Dokl. Akad. Nauk SSSR*, volume 31, pages 538–540, 1941.

[79] A.N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Dokl. Akad. Nauk SSSR*, 30(4), 1941.

[80] R. H. Kraichnan. Inertial ranges in two-dimensional turbulence. *The Physics of Fluids*, 10(7):1417–1423, 1967.

[81] I. Kukavica. On the Dissipative Scale for the Navier-Stokes Equation. *Indiana Math. J.*, 47(3):1129–1154, 1998.

[82] A. Kumar and V.R. Martinez. On well-posedness of a mildly dissipative family of active scalar equations in borderline sobolev spaces. 2021. (in preparation).

[83] L.D. Landau and E.M. Lifshitz. *Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, Volume 6*. Course of Theoretical Physics (V.6). Pergamon, 2013.

[84] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[85] A.J. Majda and A.L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[86] V.R. Martinez. Convergence analysis for a parameter estimation algorithm for the 2D Navier-Stokes equations. *Nonlinearity*, 35(5):1–29, 2022.

[87] V.R. Martinez, Z. Wang, and K. Zhao. Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. *Indiana Univ. Math. J.*, 67(4):1383–1424, 2018.

[88] V.R. Martinez and K. Zhao. Analyticity and dynamics of a Navier-Stokes-Keller-Segel system on bounded domains. *Dyn. Partial Differ. Equ.*, 14(2):125–158, 2017.

[89] J.C. Mattingly. The dissipative scale of the stochastics Navier-Stokes equation: regularization and analyticity. volume 108, pages 1157–1179. 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[90] H. Miura. Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space. *Comm. Math. Phys.*, 267(1):141–157, 2006.

[91] S.G. Resnick. *Dynamical problems in non-linear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Chicago.

[92] X. Wang. A note on long time behavior of solutions to the Boussinesq system at large Prandtl number. In *Nonlinear partial differential equations and related analysis*, volume 371 of *Contemp. Math.*, pages 315–323. Amer. Math. Soc., Providence, RI, 2005.

[93] N. Zhu, Z. Liu, V.R. Martinez, and K. Zhao. Global Cauchy problem of a system of parabolic conservation laws arising from a Keller-Segel type chemotaxis model. *SIAM J. Math. Anal.*, 50(5):5380–5425, 2018.